Locally finite logics have the density

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$A \subset Form, \quad ||\alpha||$ -length of α

We associate the density $\mu(A)$ with a subset A of formulas as:

$$\mu(A) = \lim_{n\to\infty} \frac{card\,\{\alpha \in A : ||\alpha|| = n\}}{card\,\{\alpha \in Form : ||\alpha|| = n\}}$$

if the appropriate limit exists.

If A is the set of tautologies of a given logic, then $\mu(A)$ is called *the density of truth* of this logic.

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Densities of some fragments of classical, intuitionistic and modal logics:

- $\mu(Cl^{\rightarrow}_{p,q}) \approx 51.9\%$
- $\mu(Int^{\rightarrow}_{p,q}) \approx 50.43\%$

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- $\mu(Cl^{\rightarrow,\neg}_{p}) \approx 42.3\%$
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- $\mu(Cl^{\wedge,\vee}_{p,q,\neg p,\neg q}) \approx 20.9\%$

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Negative examples:

- $\mu(Cl_p^{\leftrightarrow})$,
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Lindenbaum's algebras

$Form$ – set of all formulas in the given language with $\to$, $\wedge$, $\vee$, $\leftrightarrow$

L – propositional logic, T_L – set of theorems of the logic L.

Definition of an equivalence relation in $Form$:

$$\alpha \equiv \beta \quad \text{iff} \quad \alpha \leftrightarrow \beta \in T_L$$

for any $\alpha, \beta \in Form$.

$\equiv$ is a congruence, which means that for any unary functor $*$ and any binary functor $\odot$ it holds:

If $\alpha \equiv \beta$ then $*\alpha \equiv *\beta$,

If $\alpha \equiv \beta$ and $\gamma \equiv \delta$ then $\alpha \odot \gamma \equiv \beta \odot \delta$.

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$[\alpha]_\equiv$ – equivalence class of the relation: $[\alpha]_\equiv := \{\beta : \beta \equiv \alpha\}$.

$AL(L) := \{[\alpha]_\equiv : \alpha \in Form\}$ – Lindenbaum's algebra

Definition of order in $AL(L)$:

$$[\alpha]_\equiv \leq [\beta]_\equiv \quad \text{iff} \quad (\alpha \to \beta) \in T_L.$$

In the ordered set $(\{[\alpha]_\equiv : \alpha \in Form\}, \leq)$ there exists the supremum $[\alpha]_\equiv \cup [\beta]_\equiv = [\alpha \vee \beta]_\equiv$
and the infimum $[\alpha]_\equiv \cap [\beta]_\equiv = [\alpha \wedge \beta]_\equiv$, thus, this set forms a lattice.

In the case of classical logic, the obtained lattice is a Boolean one, whereas in the case of intuitionistic logic we get a Heyting algebra. In the cases of modal logics, we obtain modal algebras, and so on.

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Examples

- The classical logic $Cl_1^{\to,\neg}$. Lindenbaum's algebra $AL(Cl_1^{\to,\neg}) = \{[p]_\equiv, [\neg p]_\equiv, [p \to p]_\equiv, [\neg(p \to p)]_\equiv\}$ is a four-element Boolean algebra.
- The intuitionistic logic $Int_1^{\to,\neg}$. Lindenbaum's algebra $AL(Int_1^{\to,\neg}) = \{[p]_\equiv, [\neg p]_\equiv, [\neg\neg p]_\equiv, [\neg\neg p \to p]_\equiv, [p \to p]_\equiv, [\neg(p \to p)]_\equiv\}$ is a six-element Heyting algebra.

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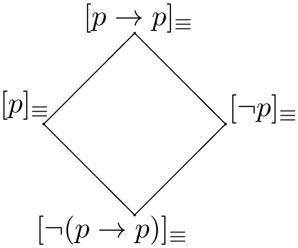
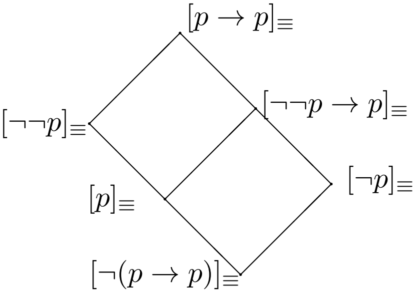


$AL(Cl_1^{\to,\neg})$

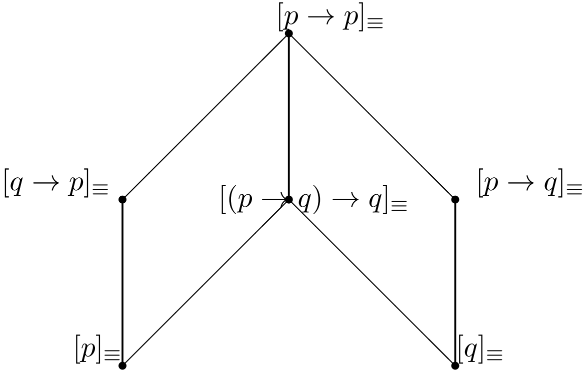


$AL(Int_1^{\to,\neg})$

- The implicational fragment of classical logic $Cl_2^{\to}$.

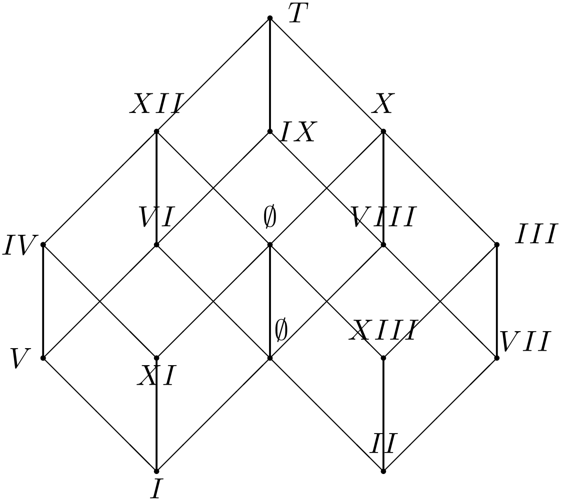
  Lindenbaum's algebra

  $AL(Cl_2^{\to}) = \{[p]_{\equiv}, [q]_{\equiv}, [p \to p]_{\equiv}, [p \to q]_{\equiv}, [q \to p]_{\equiv}, [(p \to q) \to q]_{\equiv}\}$ is a six-element upper semi-lattice.



- The intuitionistic logic $Int_2^{\rightarrow}$. Lindenbaum's algebra $AL(Int_2^{\rightarrow})$ is a fourteen-element upper semi-lattice with the following classes:

$$
\begin{array}{ll}
I = [p]_{\equiv}, & II = [q]_{\equiv} \\
III = [p \rightarrow q]_{\equiv}, & IV = [q \rightarrow p]_{\equiv} \\
V = [(p \rightarrow q) \rightarrow p]_{\equiv}, & VI = [(p \rightarrow q) \rightarrow q]_{\equiv} \\
VII = [(q \rightarrow p) \rightarrow q]_{\equiv}, & VIII = [(q \rightarrow p) \rightarrow p]_{\equiv} \\
IX = [(p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow q)]_{\equiv}, & X = [((p \rightarrow q) \rightarrow p) \rightarrow p]_{\equiv} \\
XI = [((p \rightarrow q) \rightarrow q) \rightarrow p]_{\equiv}, & XII = [((q \rightarrow p) \rightarrow q) \rightarrow q] \\
XIII = [((q \rightarrow p) \rightarrow p) \rightarrow q]_{\equiv}, & T = [p \rightarrow p]_{\equiv}
\end{array}
$$


T
XII
IX
X
VI
∅
VIII
IV
III
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XIII
V
VII
XI
II
I

Local finiteness

Logic L is locally finite (locally tabular) if in a language with a finite number of variables the number of classes of non-equivalent formulas is also finite.

That means that if the logic L is locally finite, then the Lindenbaum algebra of formulas with a finite number of variables is finite.

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Finite additivity of μ

For disjoined classes of formulas A_i such that $\mu(A_i)$ exist for each $i \leq n$, $\mu(\bigcup_{i=0}^{n} A_i)$ exists as well and

$$\mu\left(\bigcup_{i=0}^{n} A_i\right) = \sum_{i=0}^{n} \mu\left(A_i\right)$$

But μ is not countably additive:

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) \neq \sum_{i=0}^{\infty} \mu\left(A_i\right)$$

It only holds:

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) \geq \sum_{i=0}^{\infty} \mu\left(A_i\right)$$

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The Drmota-Lalley-Woods theorem

Consider a nonlinear polynomial system, defined by a set of equations $\{y_j = \Phi_j(z, y_1, ..., y_m)\}, \quad 1 \leq j \leq m$ which is a-proper, a-positive, a-irreducible and a-aperiodic. Then

- All component solutions y_i have the same radius of convergence $\rho < \infty$.
- There exist functions h_j analytic at the origin such that

$$y_j = h_j(\sqrt{1 - z/\rho}), \quad (z \to \rho^-). \tag{1}$$

- All y_j have ρ as unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form:

$$[z^n]y_j(z) \sim \rho^{-n}\left(\sum_{k\geq 1} d_k n^{-1-k/2}\right). \tag{2}$$

[6] Flajolet, P. and Sedgewick, R. *Analitic combinatorics: functional equations, rational and algebraic functions*, INRIA, Number 4103, 2001.

Application of the Drmota-Lalley-Woods theorem

Suppose we have two functions f_T and f_F enumerating the tautologies of some logic and all formulas. Suppose they have the same dominant singularity ρ and there are the suitable constants α_1, α_2, β_1, β_2 such that:

$$f_T(z) = \alpha_1 - \beta_1\sqrt{1 - z/\rho} + O(1 - z/\rho), \tag{3}$$

$$f_F(z) = \alpha_2 - \beta_2\sqrt{1 - z/\rho} + O(1 - z/\rho). \tag{4}$$

Then the *density of truth* is given by:

$$\mu(T) = \lim_{n\to\infty} \frac{[z^n]f_T(z)}{[z^n]f_F(z)} = \frac{\beta_1}{\beta_2}. \tag{5}$$

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Let $L_k^{\rightarrow}$ - locally finite logic. We assume that the functor of implication fulfils the following three very general conditions:

(i) $p \rightarrow p \in T_L$,

(ii) for any $\alpha \in Form_k^{\rightarrow}$ it holds: $\alpha \rightarrow (p \rightarrow p) \in T_L$,

(iii) for any $\alpha \in Form_k^{\rightarrow}$ it holds: $(p \rightarrow p) \rightarrow \alpha \in [\alpha]_{\equiv}$.

The conditions hold for the classical and intuitionistic implications as well as for many other implications, e.g., Łukasiewicz's and the strict implication.

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Theorem

Let L be a locally finite purely implicational logic fulfilling the conditions (i)-(iii) in language with k variables. Then the density of truth of L exists.

[7] Z.K., *On the Density of truth of locally finite logics*, Journal of Logic and Computation, Vol. 19 (6), (2009).

Proof

- L - locally finite, then Lindenbaum's algebra consists of m equivalence classes $A_1, \ldots A_m$. Let $A_m = T_L$.

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[7] Z.K., *On the Density of truth of locally finite logics*, Journal of Logic and Computation, Vol. 19 (6), (2009).

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- L - locally finite, then Lindenbaum's algebra consists of m equivalence classes $A_1, ...A_m$. Let $A_m = T_L$.

- for each A_i, we may write down a formula describing the way of creating the formulas from the given class. It is the same task as writing the appropriate truth-table.
- After translating each formula into an equation on generating functions, we obtain a system of m equations. By f_i we denote the generating function for the class A_i. Because the conditions (ii) and (iii) hold, the obtained system of equations has to look like:

$$\begin{cases} f_1 = & ... + f_m \cdot f_1 + ... \\ f_2 = & ... + f_m \cdot f_2 + ... \\ ... = & .... \\ f_m = ... + (f_1 + f_2 + ... + f_m) \cdot f_m + ... \end{cases} \tag{6}$$

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- It is easy to prove that the system (6) is a-positive, a-proper, a-irreducible. We should prove that it is a-aperiodic.

  a-aperiodicity: z (not z^2 or z^3...) is the right variable, that means for each f_j there exist three monomials z^a, z^b, and z^c such that $b-a$ and $c-a$ are relatively prime. Then for each generating function $f_j(z)=\sum_{n=0}^{\infty} c_{jn} z^n$ there is some n_0 such that for all $n>n_0$ it holds $c_{jn}\neq 0$.

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The formula $p \to p$ is the shortest tautology (of the length 2).

From (ii) we conclude that in the class T_L there are formulas of each length greater than or equal to 2. Then in the expansion $f_m(z) = \sum_{n=2}^{\infty} c_{mn} z^n$ the coefficients $c_{mn} \neq 0$ for $n \geq 2$. Next, from (iii) we conclude that if the shortest formula from A_j has, for instance, the length l, then in the class A_j there are formulas of each length $\geq l + 2$. Hence we have $f_j(z) = \sum_{n=l}^{\infty} c_{jn} z^n$, and $c_{jn} \neq 0$ for $n = l$ and $n \geq l + 2$. That means that the system of equations (6) is a-aperiodic.

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Corollary

Let L be a locally finite logic with implication and other functors as well. Then the density $\mu(L)$ exists.

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Application

Theorem

Classical logic is locally finite and this fact does not depend on the chosen set of functors.

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[Diego-Popiel] The implicational fragment of intuitionistic logic is locally finite. The implicational-negational fragment of intuitionistic logic is locally finite.

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Conjuncture from [8] and [1]

$$\lim_{k\to\infty} \frac{\mu(Int_k^{\rightarrow})}{\mu(Cl_k^{\rightarrow})} = 1$$

assuming that the densities exist.

[8] Moczurad M., Tyszkiewicz J., Zaionc M. *Statistical properties of simple types*, Mathematical Structures in Computer Science, vol 10, 2000, pp 575-594.

Result from [9].

$$\lim_{k\to\infty} \frac{\mu^-(Int_k^{\to})}{\mu(Cl_k^{\to})} = 1$$

where $\mu^-(Int_k^{\to}) = \liminf_{n\to\infty} \frac{|Int_k^{\to} \cap Form_k^n|}{|Form_k^n|}$ and $Form_k^n$ – set of implicational formulas of length n with k variables.

[9] Fournier H., Gardy D., Genitrini A., Zaionc M. *Classical and intuitionistic logic are asymptotically identical*, Lecture Notes in Computer Science 4646, pp. 177-193.

A strengthening

Theorem

The densities $\mu(Cl_k^{\rightarrow})$ *and* $\mu(Int_k^{\rightarrow})$ *of the implicational fragments of classical and intuitionistic logics exist and it holds;*

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Locally finite modal logics with implication

A logic $L \in NEXT(\mathbf{K4})$ is locally finite iff L is of finite depth.

Let us consider the family $\mathbf{K4} \oplus \mathbf{bd_n}$ for each $n \geq 1$, where

$$\begin{aligned} \mathbf{bd_1} &= \Diamond\Box p_1 \to p_1, \\ \mathbf{bd_{n+1}} &= \Diamond(\Box p_{n+1} \wedge \neg \mathbf{bd_n}) \to p_{n+1}. \end{aligned}$$

The logics $\mathbf{K4} \oplus \mathbf{bd_n}$ for each $n \geq 1$ have finite depth.

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Let $L \in NEXT(\mathbf{K4} \oplus \mathbf{bd_n})$ for any $n \geq 1$. Then its density of truth exists.

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Why $\mu(Cl_1^{\leftrightarrow})$ does not exist?

Lindenbaum's algebra is a two-element Boolean algebra: $AL(Cl_1^{\leftrightarrow}) = \{[p \leftrightarrow p]_{\equiv}, [p]_{\equiv}\}$. In this fragment of classical logic, the functor of implication is not definable and moreover the length of each tautology is an even number, whereas the length of each non-tautology is odd, see [10].

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$$f_T(z) = 1z^2 + 5z^4 + 42z^6 + ...$$

The explicit formula for f_T:

$$f_T(z) = \frac{1}{4}\left(2 - \sqrt{1-4z} - \sqrt{1+4z}\right).$$

There are two singularities $z_1 = \frac{1}{4}$ and $z_2 = -\frac{1}{4}$. Analogously $Cl_2^{\leftrightarrow}$ and $Int_2^{\leftrightarrow}$. Also: $Cl_2^{\leftrightarrow,\neg}$ and $Int_2^{\leftrightarrow,\neg}$.

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Question

What can we say about logics with implication fulfilling the conditions (i)-(iii) which are not locally finite?

Do they have the density of truth?

Example: $Int_p^{\to,\neg,\vee}$ or $Int_{p,\bot}^{\to,\vee}$

$$\alpha^0 = \neg(p \to p), \quad \alpha^1 = p, \quad \alpha^2 = \neg p,$$
$$\alpha^{2n+1} = \alpha^{2n} \vee \alpha^{2n-1}, \quad \alpha^{2n+2} = \alpha^{2n} \to \alpha^{2n-1} \quad \text{for } n \geq 1$$
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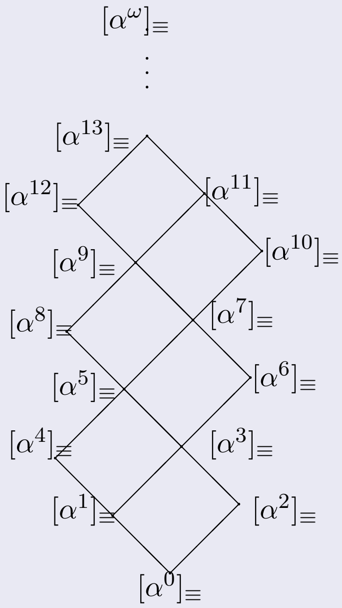
The Rieger-Nishimura lattice



Lemma

The density of truth of $Int_p^{\to,\neg,\vee}$ exists and it is estimated as follows:

$$0.7068 \leq \mu(Int_p^{\to,\neg,\vee}) \leq 0.709011$$

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