TACL
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On Kripke incomplete logics containing KTB

Zofia Kostrzycka
Opole University of Technology

Brouwerian logic **KTB**

Axioms CL and

$$K := \Box(p \to q) \to (\Box p \to \Box q)$$
$$T := \Box p \to p$$
$$B := p \to \Box\Diamond p$$

and rules: (MP), (Sub) i (RG).

Definition 1. *A logic* $\mathbf{L}$ *is* *Kripke complete*, *if there is a class* $\mathcal{C}$ *of Kripke frames, such that:*

1. *for every formula* $\psi \in \mathbf{L}$ *and any frame* $\mathfrak{F} \in \mathcal{C}$ *we have* $\mathfrak{F} \models \psi$.

2. *for every formula* $\psi \notin \mathbf{L}$, *there is a Kripke frame* $\mathfrak{G} \in \mathcal{C}$ *such that* $\mathfrak{G} \not\models \psi$.

Kripke frames for $\mathbf{KTB}$

Saul Kripke, *Semantical analysis of modal logic* , 1963:

$\mathfrak{F} = \langle W, R \rangle$ where W -nonempty set and R - reflexive and symmetric relation on W.

Extensions of $\mathbf{KTB}$

Ivo Thomas defined in 1964:

$\mathbf{T_n} = \mathbf{KTB} \oplus (4_n)$, where

$$(4_n) \qquad \Box^n p \rightarrow \Box^{n+1} p$$

$$(tran_n) \quad \forall_{x,y} \, (\text{if } \; xR^{n+1}y \; \text{ then } \; xR^n y)$$

$$\mathbf{KTB} \subset ... \subset \mathbf{T_{n+1}} \subset \mathbf{T_n} \subset ... \subset \mathbf{T_2} \subset \mathbf{T_1} = \mathbf{S5}.$$

PROBLEM 1

Miyazaki [1] constructed one Kripke incomplete logic in $NEXT(\mathbf{T_2})$ and continuum Kripke incomplete logics in $NEXT(\mathbf{T_5})$.

Question: Is there a continuum of Kripke incomplete logics in $NEXT(\mathbf{T_2})$?

[1] Y. Miyazaki, *Kripke incomplete logics containing KTB*, Studia Logica, 85, (2007), 311-326.

[2] Z. Kostrzycka, *On the existence of a continuum of logics in $NEXT(KTB \oplus \Box^2 p \to \Box^3 p)$*, Bulletin of the Section of Logic, Vol.36/1, (2007), 1-7.

A sequence of non-equivalent formulas

Denote $\alpha := p \wedge \neg\Diamond\Box p$.

Definition 2.

$$A_1 := \neg p \wedge \Box\neg\alpha$$

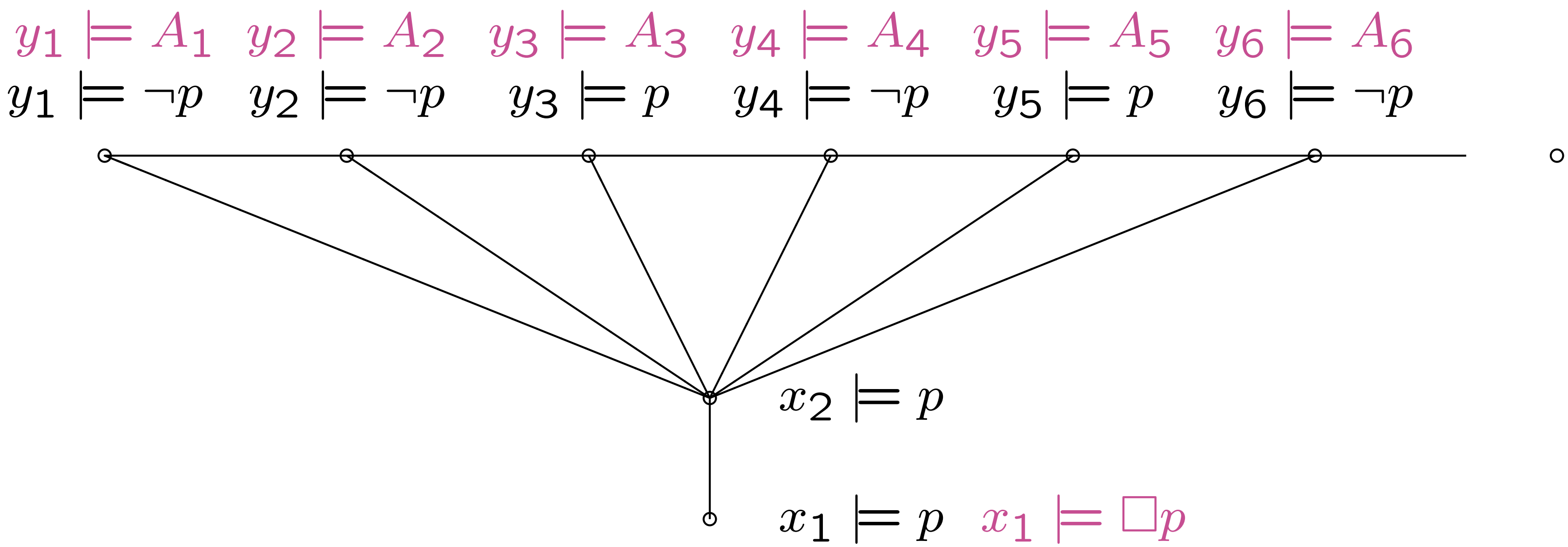
$$A_2 := \neg p \wedge \neg A_1 \wedge \Diamond A_1$$

$$A_3 := \alpha \wedge \Diamond A_2 \wedge \neg\Diamond A_1$$

For $n \geq 2$:

$$\begin{aligned} A_{2n} &:= \neg p \wedge \Diamond A_{2n-1} \wedge \neg A_{2n-2} \\ A_{2n+1} &:= \alpha \wedge \Diamond A_{2n} \wedge \neg A_{2n-1} \end{aligned}$$

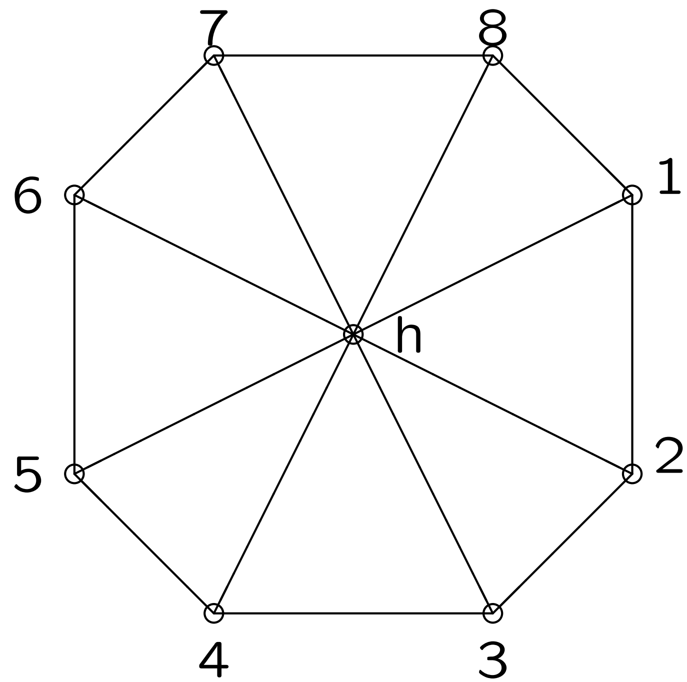
Theorem 3. *The formulas $\{A_i\}$, $i \geq 1$ are non-equivalent in the logic $\mathbf{T_2}$.*



For any $i \geq 1$ and for any $x \in W$ the following holds:

$$x \models A_i \quad \text{iff} \quad x = y_i$$

Wheel frames



A diagram of the $\mathfrak{W}_8$

Theorem 4. *$\mathfrak{W}_m$ is reducible to $\mathfrak{W}_n$ iff m is divisible by n, for $n \geq 5$.*

Let:

$$\beta := \neg\Box p \wedge \Diamond\Box p$$
$$\gamma := \beta \wedge \Diamond A_1 \wedge \neg\Diamond A_2$$
$$\varepsilon := \beta \wedge \neg\Diamond A_1 \wedge \neg\Diamond A_2$$

$$C_k := \Box^2[A_{k-1} \to \Diamond A_k], \text{ for } k > 2$$
$$D_k := \Box^2[(A_k \wedge \neg\Diamond A_{k+1}) \to \Diamond\varepsilon],$$
$$E := \Box^2(\Box p \to \Diamond\gamma)$$

$$G_k := (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E) \to \Diamond^2 A_k.$$

Lemma 5. *Let $k \geq 5$ and k- odd number.*

$\mathfrak{W}_i \not\models G_k$ *iff i is divisible by $k+2$.*

$Prim := \{n \in \omega : n + 2 \text{ is prime}, n \geq 5\}$, $X \subset Prim$,

$L_X := \mathbf{T_2} \oplus \{G_k : k \in X\}$-uncountable family in $NEXT(\mathbf{T_2})$

Kripke incomplete extensions of L_X

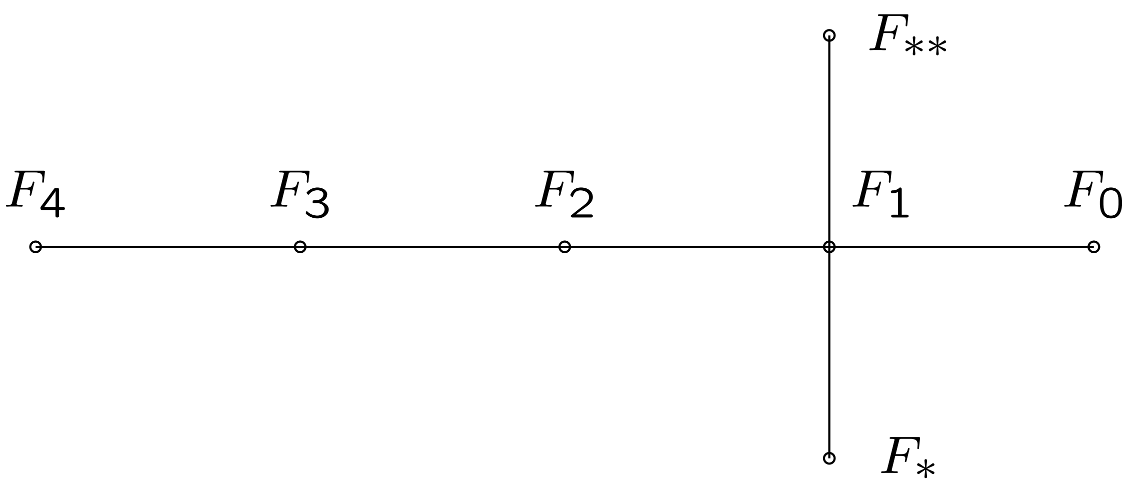
Modification of Miyazaki's constructions.

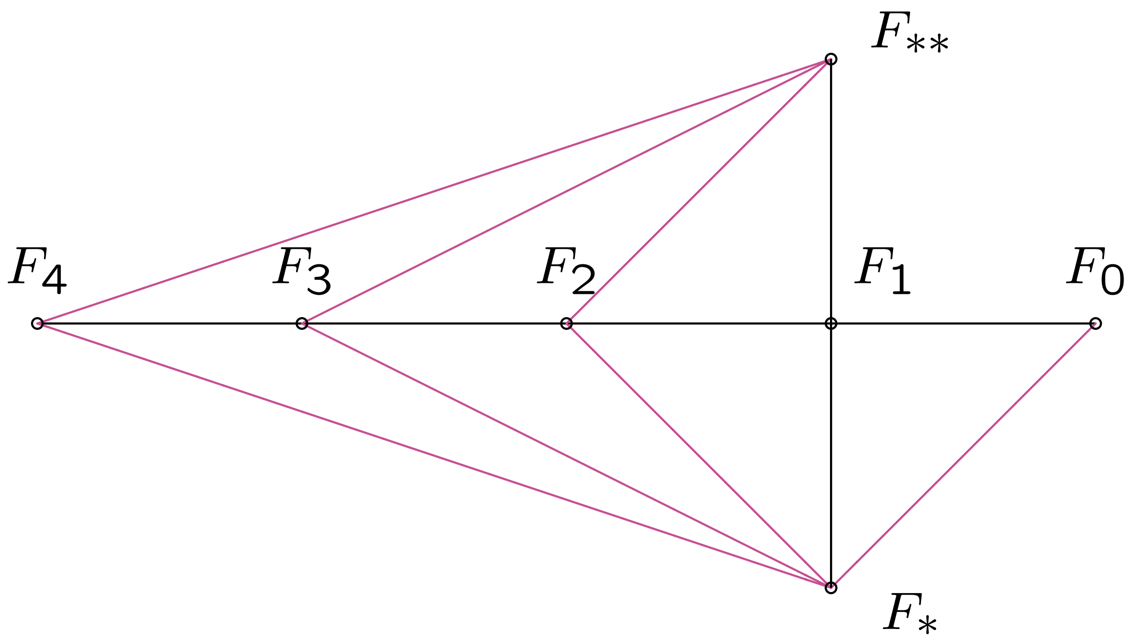
Exclusive formulas:

$$
\begin{aligned}
F_* &:= p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \\
F_{**} &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \\
F_0 &:= \neg p_* \wedge p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \\
F_1 &:= \neg p_* \wedge \neg p_0 \wedge p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \\
F_2 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4 \\
F_3 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4 \\
F_4 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge p_4
\end{aligned}
$$

$$
\begin{aligned}
Q \quad := \quad & \{F_1 \wedge \Diamond F_* \wedge \Diamond(F_{**} \wedge \neg\Diamond F_0) \wedge \Diamond(F_0 \wedge \neg\Diamond F_3 \wedge \neg\Diamond F_4) \wedge \\
& \wedge \Diamond(F_2 \wedge \Diamond(F_3 \wedge \Diamond F_4) \wedge \neg\Diamond F_0 \wedge \neg\Diamond F_4) \wedge \neg\Diamond F_3 \wedge \neg\Diamond F_4\} \rightarrow \\
& \rightarrow \{\Diamond(F_* \wedge \Diamond F_0 \wedge \Diamond(F_2 \wedge \Diamond(F_{**} \wedge \Diamond F_3 \wedge \Diamond F_4)) \wedge \Diamond F_3 \wedge \Diamond F_4)\},
\end{aligned}
$$

The role of the formula Q:





F_{**}
F_4
F_3
F_2
F_1
F_0
F_*

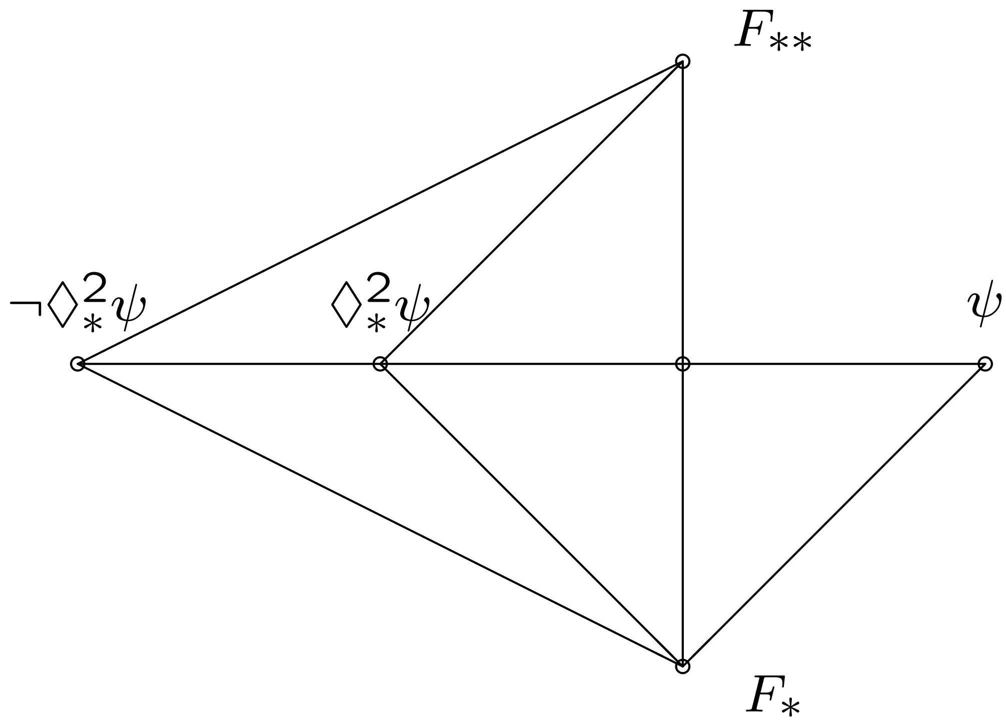
$$
\begin{aligned}
R_n \quad := \quad & \{F_* \wedge \Diamond(F_0 \wedge \neg\Diamond F_2 \wedge \neg\Diamond F_3 \wedge \neg\Diamond F_4 \wedge [\neg\Diamond_*^{n-1} \wedge \Diamond_*^{n}](F_1 \wedge \\
& \wedge\Diamond(F_2 \wedge \Diamond(F_3 \wedge \Diamond(F_4 \wedge \Diamond(F_{**} \wedge \Diamond F_1 \wedge \Diamond F_2 \wedge \Diamond F_3))))) \\
& \wedge\Diamond(F_1 \wedge \neg\Diamond F_3 \wedge \neg\Diamond F_4) \wedge \Diamond F_2 \wedge \Diamond F_3 \wedge \Diamond F_4 \wedge \neg\Diamond^2(F_0 \wedge \Diamond F_{**}) \\
& \rightarrow \Diamond\{F_4 \wedge \Diamond([\neg\Diamond_*^{n+3} \wedge \Diamond_*^{n+4}]F_0 \wedge \Diamond F_* \wedge \Diamond F_{**})\},
\end{aligned}
$$

where

$$\Diamond_*^0\psi := \psi,$$
$$\Diamond_*^1\psi := \Diamond(\neg F_* \wedge \neg F_{**} \wedge \psi),$$
$$\Diamond_*^k\psi := \Diamond(\neg F_* \wedge \neg F_{**} \wedge \Diamond_*^{k-1}\psi)$$

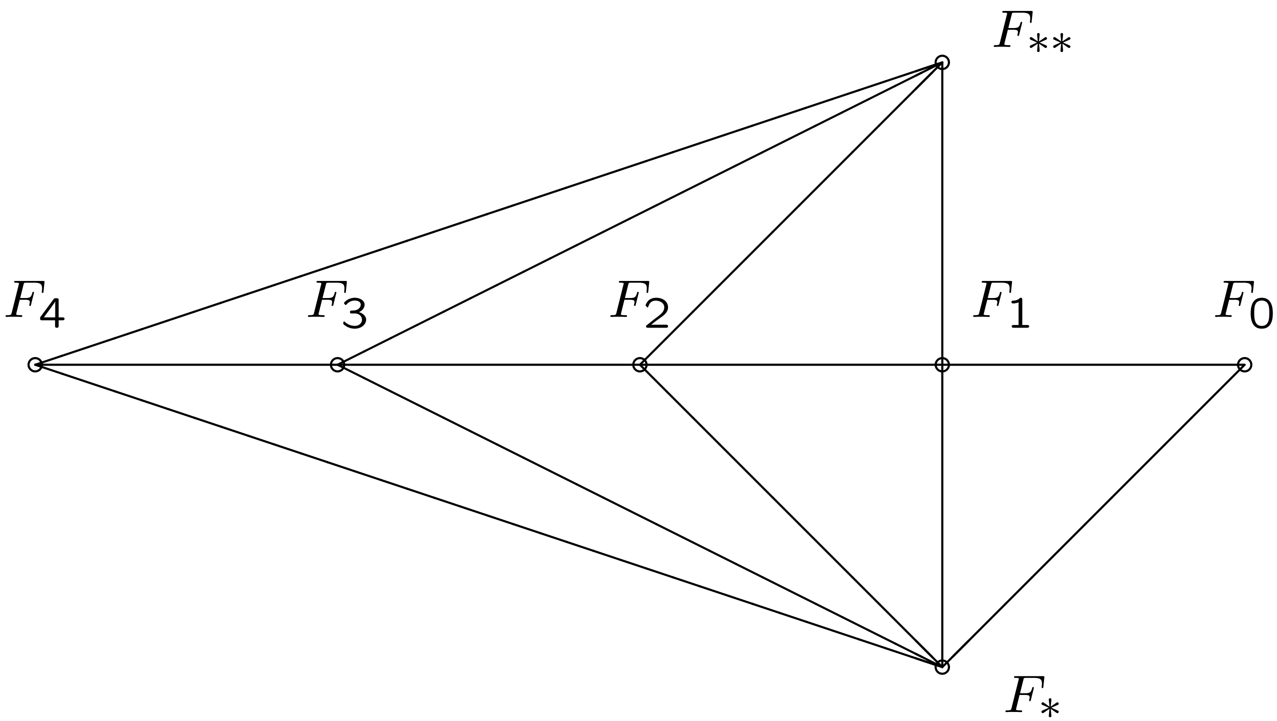
and

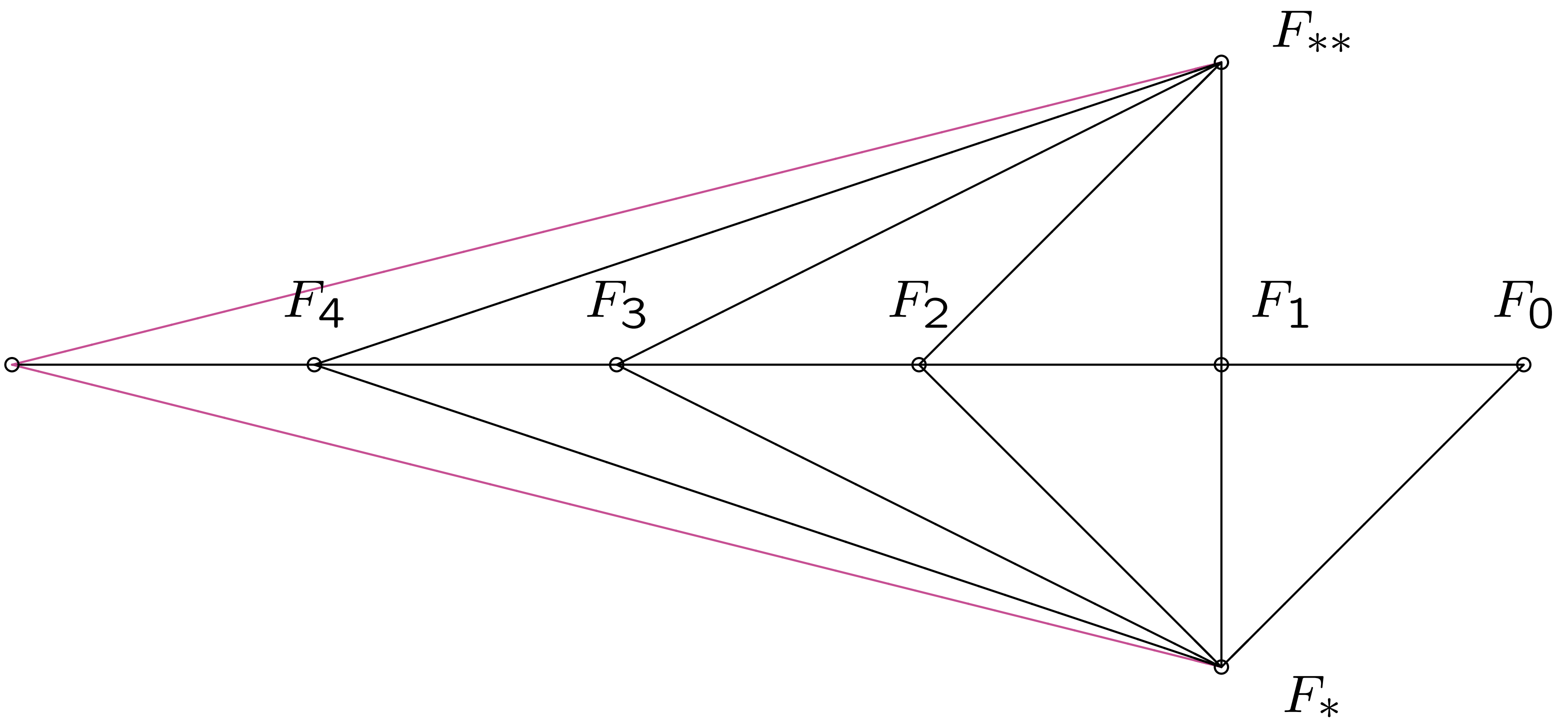
$$[\neg\Diamond_*^{n-1} \wedge \Diamond_*^{n}]\psi := \neg\Diamond_*^{n-1}\psi \wedge \Diamond_*^{n}\psi.$$



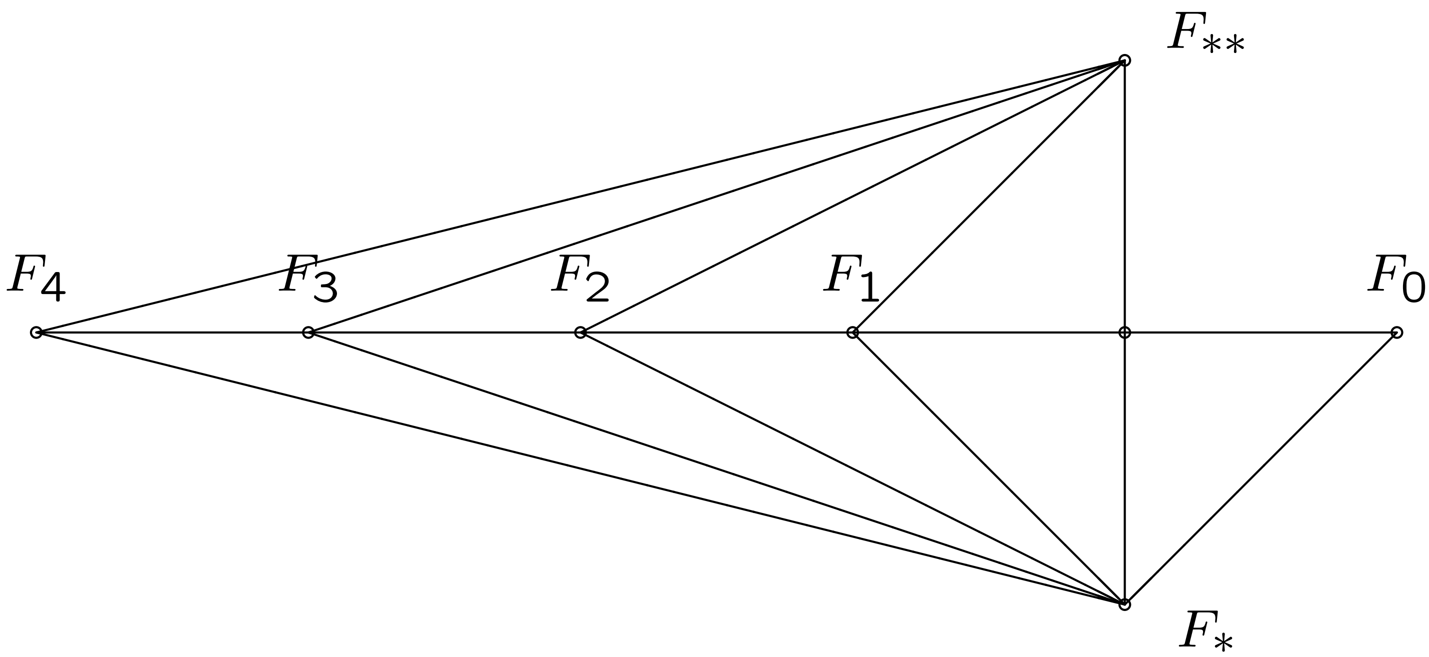
$$\Diamond_*^2\psi := \Diamond(\neg F_* \wedge \neg F_{**} \wedge \Diamond(\neg F_* \wedge \neg F_{**} \wedge \psi))$$

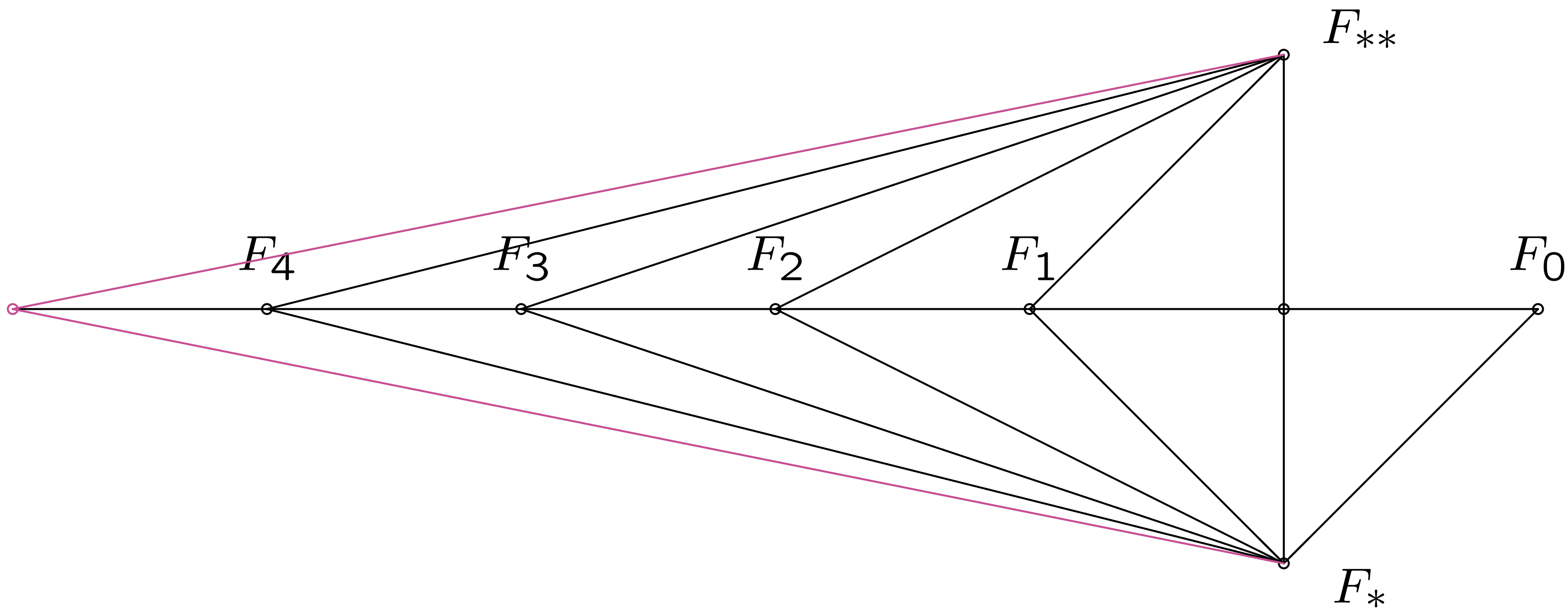
The role of the formula R_1:




F_{**}
F_4
F_3
F_2
F_1
F_0
F_*

The role of the formula R_2:




F_{**}
F_4
F_3
F_2
F_1
F_0
F_*

Definition 6. $L'_X := \mathbf{T_2} \oplus \{G_k : k \in X\} \oplus Q \oplus \{R_n : n \geq 1\}$

Theorem 7. *For each X the logic L'_X is Kripke incomplete.*

Theorem 8. *The family of logics L'_X is an uncountable family of Kripke incomplete logics in $NEXT(T_2)$.*

[3] Kostrzycka Z., *On non-compact logics in NEXT(KTB)*, Math. Log. Quart. 54, No. 6, (2008), 582-589.

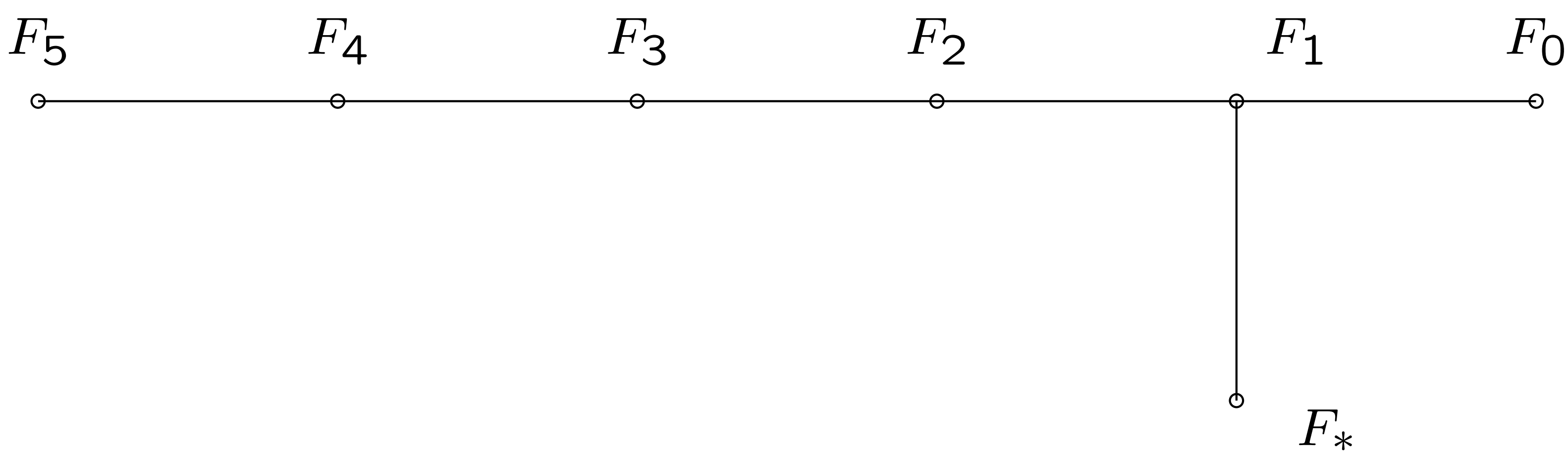
PROBLEM 2: Is there a finitely axiomatizable extension of $\mathbf{T_2}$ which is Kripke incomplete?

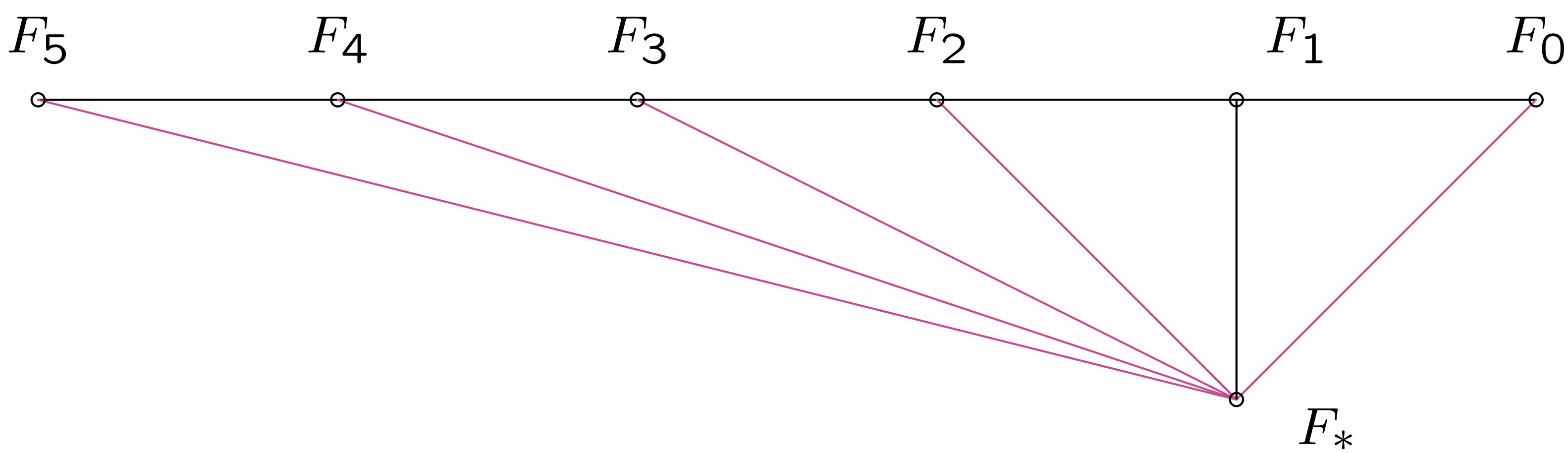
Axioms for L_*

$F_*, F_0, F_1, F_2, F_3, F_4, F_5$ - exclusive formulas:

$$Q' := \{F_1 \wedge \Diamond F_* \wedge \Diamond(F_0 \wedge \neg\Diamond F_2 \wedge \neg\Diamond F_3 \wedge \neg\Diamond F_4) \wedge$$
$$\wedge\Diamond(F_2 \wedge \Diamond(F_3 \wedge \Diamond(F_4 \wedge \Diamond F_5)) \wedge \neg\Diamond F_4 \wedge \neg\Diamond F_5) \wedge \neg\Diamond F_3\}$$
$$\rightarrow \Diamond(F_* \wedge \Diamond F_0 \wedge \Diamond F_2 \wedge \Diamond F_3 \wedge \Diamond F_4 \wedge \Diamond F_5).$$

The role of Q':

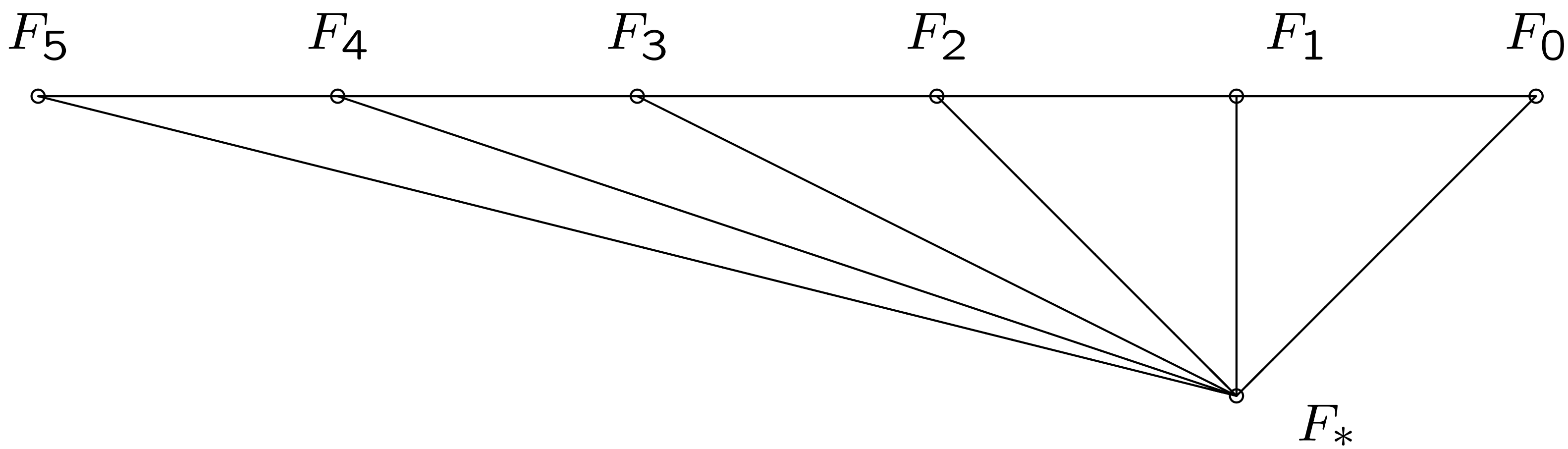


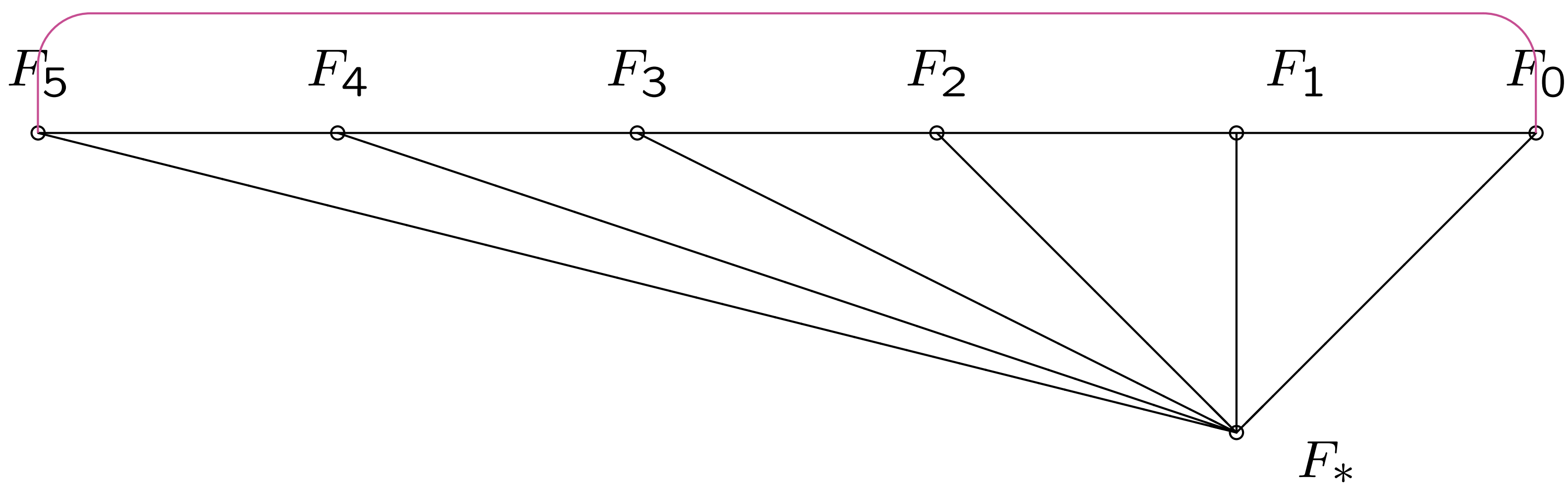


F_5
F_4
F_3
F_2
F_1
F_0
F_*

$$
\begin{aligned}
K \quad := \quad & \{F_5 \wedge \Diamond[F_4 \wedge \Diamond(F_3 \wedge \Diamond(F_2 \wedge \Diamond(F_1 \wedge \Diamond F_0)))] \wedge \Diamond F_* \\
& \wedge \bigwedge_{i=0}^{5} \Box^2(F_i \rightarrow \Box p) \wedge \Box^2 \left((p \wedge F_*) \rightarrow \bigwedge_{i=0}^{5} \Diamond F_i \right) \wedge \\
& \wedge \Box^2 \left(F_* \vee \bigvee_{i=0}^{5} F_i \right) \wedge \Box^2[\Box(F_5 \vee (F_* \wedge p)) \rightarrow \Diamond(F_5 \wedge \Diamond F_4)] \wedge \\
& \wedge \bigwedge_{i=0}^{4} \Box^2[\Box(F_i \vee (F_* \wedge p)) \rightarrow \Diamond(F_i \wedge \Diamond F_{i+1})]\} \rightarrow \Diamond F_0
\end{aligned}
$$

The role of K:




F_5
F_4
F_3
F_2
F_1
F_0
F_*

$$P := \{r \wedge \bigwedge_{i=1}^{3} (A_i \wedge B_i \wedge C_i)\} \to \Diamond^2\{r \wedge \Box(r \to (q_1 \vee q_2 \vee q_3))\},$$

where

$$A_i := \Box^2(q_i \to r), \quad B_i := \Box^2(r \to \Diamond q_i), \quad \text{for} \quad i = 1, 2, 3$$

$$C_1 := \Box^2\neg(q_2 \wedge q_3), \quad C_2 := \Box^2\neg(q_1 \wedge q_3), \quad C_3 := \Box^2\neg(q_1 \wedge q_2).$$

Theorem 9. *The logic* $L_* = \mathbf{T_2} \oplus K \oplus Q' \oplus P$ *is Kripke incomplete.*

[4] Kostrzycka Z., *On a finitely axiomatizable Kripke incomplete logic containing KTB*, Journal of Logic and Computation.

Proof

We find a formula ψ such that $\psi \notin L_*$
and
for any Kripke frame $\mathfrak{F}$ the following implication holds:

if $\mathfrak{F} \models L_*$, then $\mathfrak{F} \models \psi$.

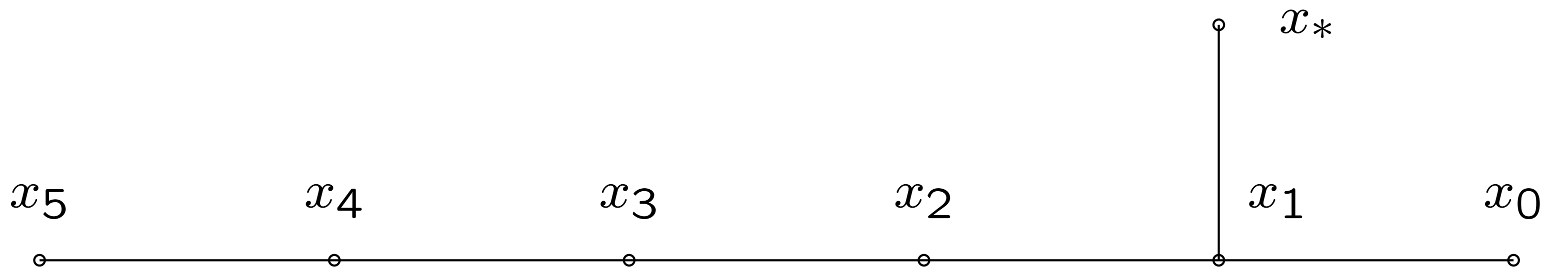
Formuła ψ

$$H_* := \neg s_0 \wedge \neg s_1 \wedge \neg s_2 \wedge \neg s_3 \wedge \neg s_4,$$
$$H_0 := \Box\neg s_0 \wedge \neg s_1 \wedge s_2 \wedge s_3 \wedge s_4,$$
$$H_1 := \neg s_0 \wedge \Box\neg s_1 \wedge \neg s_2 \wedge s_3 \wedge s_4,$$
$$H_2 := s_0 \wedge \neg s_1 \wedge \Box\neg s_2 \wedge \neg s_3 \wedge s_4,$$
$$H_3 := s_0 \wedge s_1 \wedge \neg s_2 \wedge \Box\neg s_3 \wedge \neg s_4,$$
$$H_4 := s_0 \wedge s_1 \wedge \neg s_2 \wedge \neg s_3 \wedge \Box\neg s_4,$$
$$H_5 := \neg s_0 \wedge s_1 \wedge \neg s_2 \wedge s_3 \wedge \neg s_4,$$

$$\psi := \neg\{H_5 \wedge \Diamond[H_4 \wedge \Diamond(H_3 \wedge \Diamond(H_2 \wedge \Diamond(H_1 \wedge \Diamond H_0 \wedge \Diamond H_*)))]\}.$$

Suppose that there is a Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \models L_*$ and $\mathfrak{F} \not\models \psi$.

Then the structure $\mathfrak{F}$ consists of at least seven different points $x_*, x_0, x_1, x_2, x_3, x_4, x_5$ such that: $x_1 R x_*$, and $x_i R x_j$ iff $|i - j| \leq 1$ for $i, j = 0, ..., 4$ i $x_4 R x_5$.

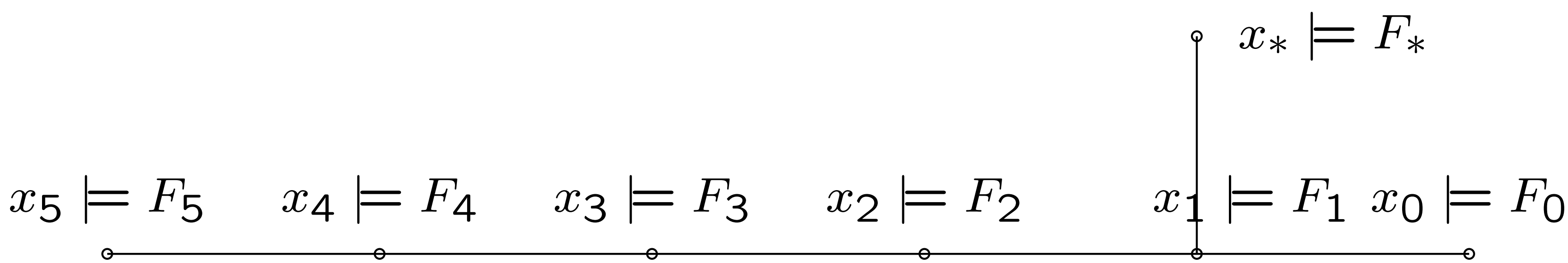


We define a valuation for the variables $p_0, ..., p_5, p_*$:

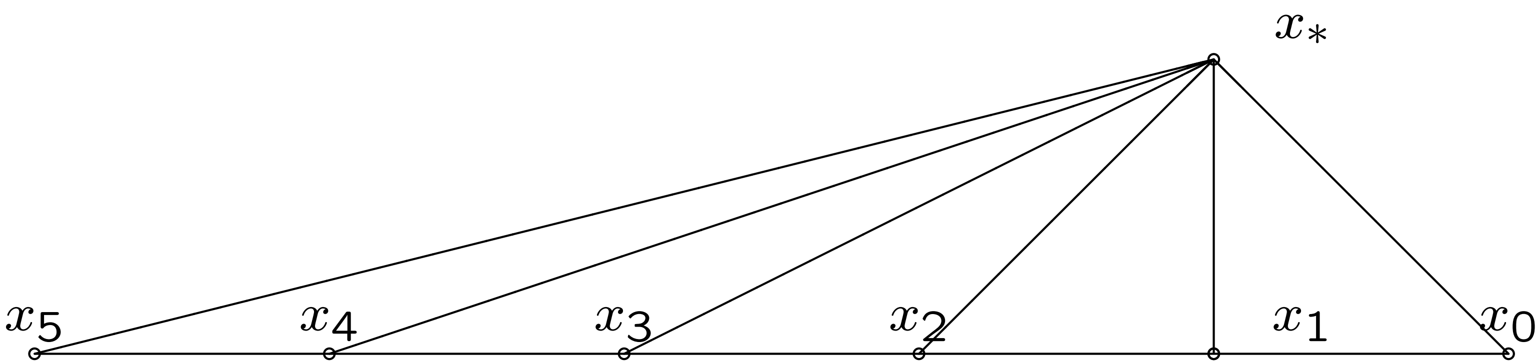
$$V(p_i) = \{x_i\} \quad \text{for} \quad i = 0, ..., 5, \quad \text{and} \quad V(p_*) = \{x_*\}.$$

That gives us:

$$V(F_i) = \{x_i\} \quad \text{for} \quad i = 0, ..., 5, \quad \text{and} \quad V(F_*) = \{x_*\}.$$



The formula Q' has to be true under that valuation, hence it must hold: $x_* R x_j$, for $j = 0, 2, 3, 4, 5$.

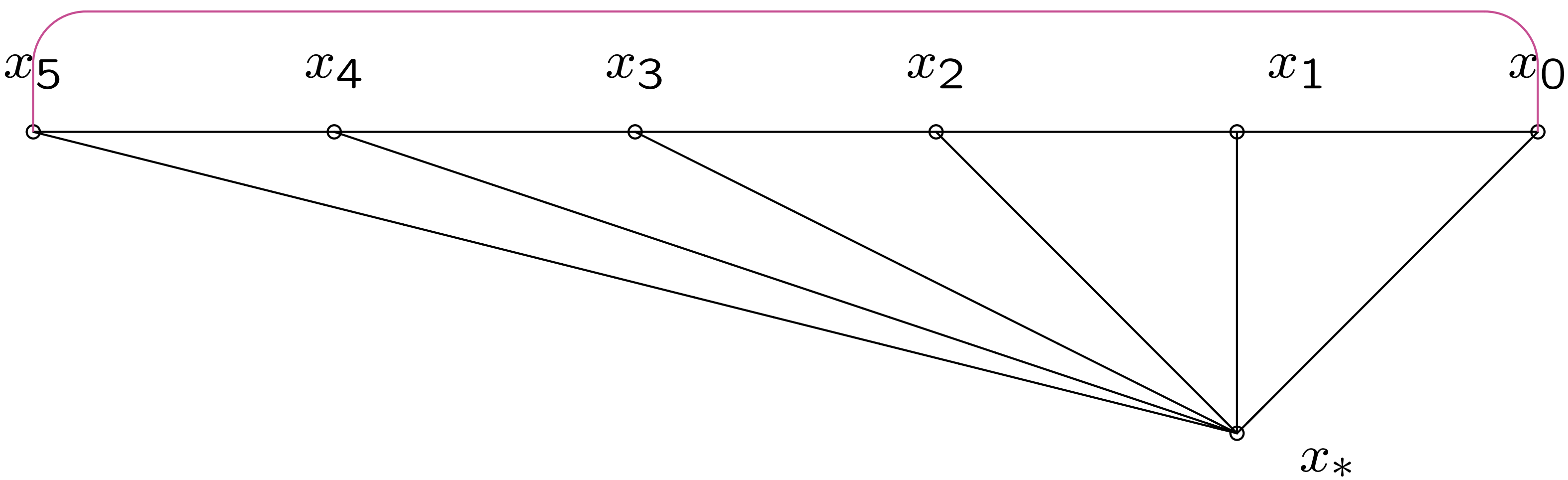


Let us consider a new valuation defined on the obtained frame:

$$x_* \models p_*, \quad x_i \models p_i, \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5$$

For such valuation we obtain:

$$x_* \models F_* \wedge p \quad \text{iff} \quad x = x_*$$
$$x \models F_0 \quad \text{iff} \quad x = x_0$$
$$x \models F_5 \quad \text{iff} \quad x = x_5$$

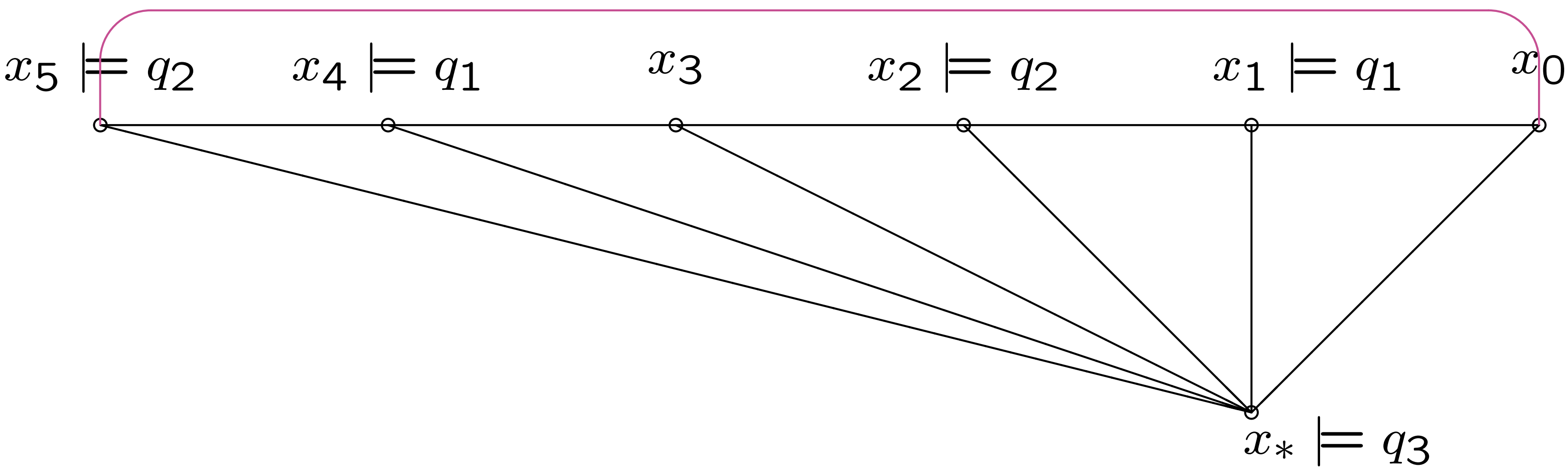


$$
\begin{aligned}
P &:= \{r \wedge \bigwedge_{i=1}^{3} (A_i \wedge B_i \wedge C_i)\} \to \Diamond^2\{r \wedge \Box(r \to (q_1 \vee q_2 \vee q_3))\}, \\
& \quad \text{where} \\
A_i &:= \Box^2(q_i \to r), \quad B_i := \Box^2(r \to \Diamond q_i), \quad \text{for} \quad i = 1, 2, 3 \\
C_1 &:= \Box^2\neg(q_2 \wedge q_3), \quad C_2 := \Box^2\neg(q_1 \wedge q_3), \quad C_3 := \Box^2\neg(q_1 \wedge q_2).
\end{aligned}
$$

Formula P is false with the following valuation:

$V_*(r) = \{x_*, x_0, ..., x_5\}$, $V_*(q_1) = \{x_1, x_4\}$, $V_*(q_2) = \{x_2, x_5\}$

$$V_*(q_3) = \{x_*\}.$$



We take x_3. It holds: $x_3 \models r$ and $x_3 \models A_i \wedge B_i \wedge C_i$ for $i = 1, 2, 3$. However $x_{3n} \not\models q_1 \vee q_2 \vee q_3$ for $n = 0, 1$, and then $x_3 \not\models \Diamond^2\{r \wedge \Box(r \to (q_1 \vee q_2 \vee q_3))\}$.

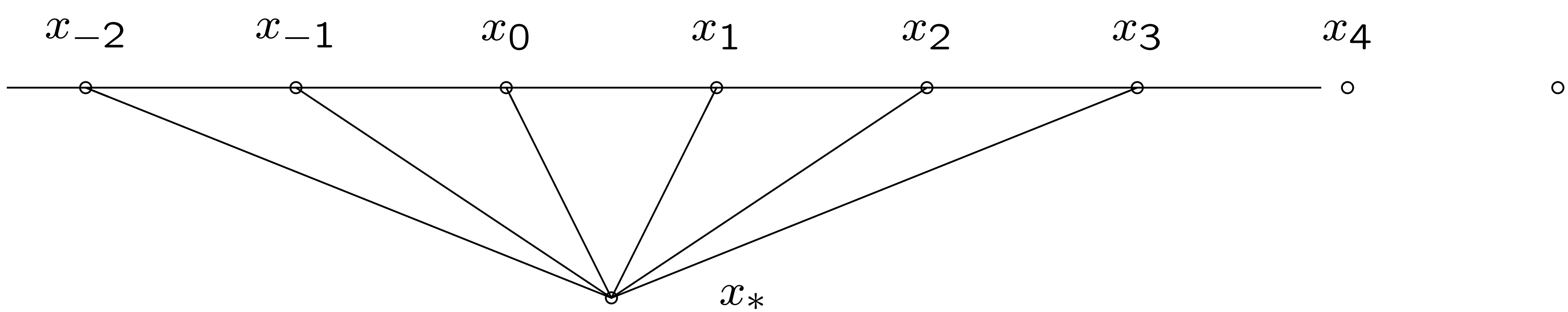
Hence: $x_3 \not\models P$.

We use a general frame to show that $\psi \notin L_*$.

Define:

$\mathfrak{G} = \langle W, R, T \rangle$ where:

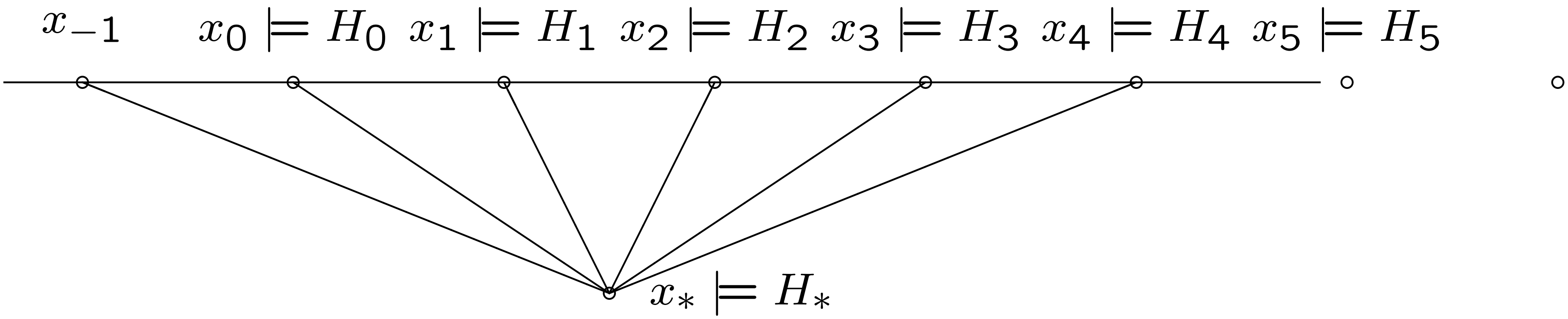
$$
\begin{aligned}
W &:= \{x_*\} \cup \{x_i, i \in \mathbb{Z}\}, \\
R &:= \{(x_*, x_i) \text{ for each } i \in \mathbb{Z}, \} \cup \\
&\quad \cup \{(x_i, x_j) \text{ iff } |i - j| \leq 1; \text{ for any } i, j \in \mathbb{Z}\}, \\
T &:= \{X \subset W : X \text{ is finite or } W \setminus X \text{ is finite}\}.
\end{aligned}
$$



$\mathfrak{G} \models P, Q', K.$

Define a valuation:

$$V(s_0) = \{x_2, x_3, x_4\},\ \ V(s_1) = \{x_3, x_4, x_5\},\ \ V(s_2) = \{x_0, x_4, x_5\},$$
$$V(s_3) = \{x_0, x_1, x_5\},\ \ V(s_4) = \{x_0, x_1, x_2\}.$$



Then for

$$\psi := \neg\{H_5 \wedge \lozenge[H_4 \wedge \lozenge(H_3 \wedge \lozenge(H_2 \wedge \lozenge(H_1 \wedge \lozenge H_0 \wedge \lozenge H_*)))]\}.$$

we obtain $\mathfrak{G} \not\models \psi$.

THANK YOU