Computational Logic and Applications
KRAKÓW 2008

On density of truth of infinite logic

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By locally infinite logic, we mean a logic, which in some language with a finite number of variables, has infinitely many classes of non-equivalent formulas.

Examples:

$Int_k^{\rightarrow}$, $Cl_k^{\rightarrow}$, $Int_k^{\rightarrow,\bot}$ $Cl_k^{\rightarrow,\bot}$ – locally finite

$Cl_k^{\rightarrow,\vee,\bot}$ – locally finite

$Int_k^{\rightarrow,\vee,\bot}$ – locally infinite

L- given logic

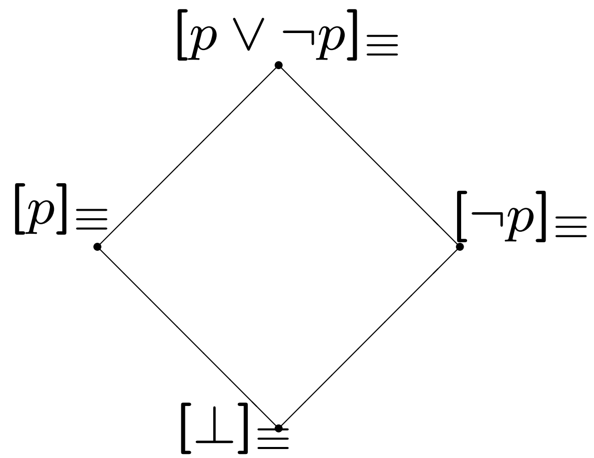
Definition 1. $\varphi \equiv_L \psi$ *if both* $\varphi \to \psi \in Taut_L$ *and* $\psi \to \varphi \in Taut_L$.

Definition 2. $L/_{\equiv} = \{[\alpha]_{\equiv}, \alpha \in Form\}$

Definition 3. *The order of classes* $[\alpha]_{\equiv}$ *is defined as*

$[\alpha]_{\equiv} \leq [\beta]_{\equiv}$ *iff* $\alpha \to \beta \in Taut_L$.

System $Cl_1^{\rightarrow,\vee,\bot}$



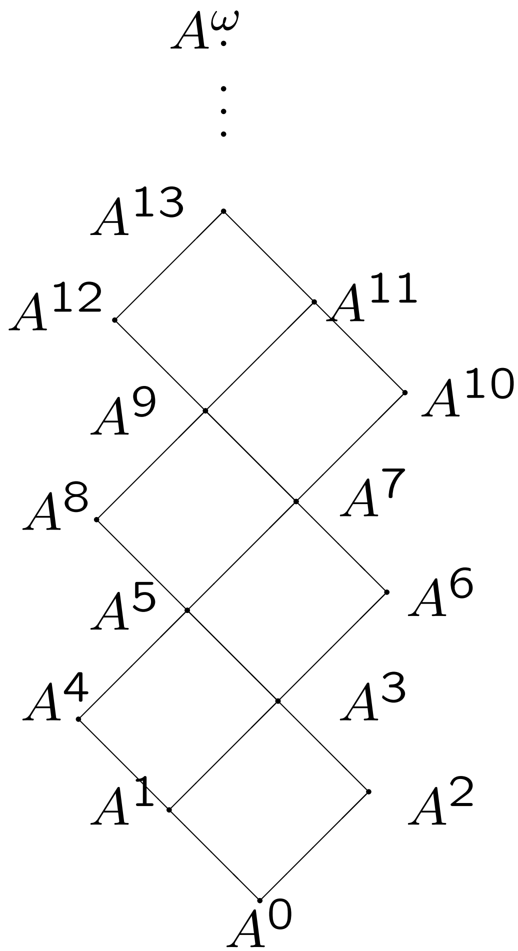
where $\neg p := p \rightarrow \bot$.

System $Int_1^{\to,\vee,\bot}$

$$\alpha^0 = \bot$$
$$\alpha^1 = p$$
$$\alpha^2 = p \to \bot$$
$$\alpha^{2n+1} = \alpha^{2n} \vee \alpha^{2n-1}$$
$$\alpha^{2n+2} = \alpha^{2n} \to \alpha^{2n-1}$$

for $n \geq 1$

Rieger - Nishimura lattice



$$[\alpha^k]_\equiv = A^k$$

Intuitionistic logic - motivation

To exclude non-constructive proofs.

1. A proof of $\alpha \wedge \beta$ consists of a proof of α and a proof of β.

2. A proof of $\alpha \vee \beta$ is given by presenting either a proof of α or a proof of β.

3. A proof of $\alpha \longrightarrow \beta$ is a construction which, giving a proof of α, returns a proof of β.

4. $\perp$ has no proof and a proof of $\neg\alpha$ is a construction which, given a proof of α, would return a proof of $\perp$.

States of knowledge

Our knowledge is developing discretely, passing from one state to another.

Let $x_1 R x_2$.

If $x_1 : \alpha = 1$, then $x_2 : \alpha = 1$.

But it is possible:

$x_1 : \alpha = 0$ and $x_2 : \alpha = 1$.

Kripke frames and models for Int

Definition 4. *An intuitionistic Kripke frame is a pair $\mathfrak{F} = \langle W, R \rangle$ consisting of non-empty set W and a partial order R on W. That means that R is reflexive, transitive and antisymmetric.*

Elements of W are called the points, and xRy is read 'y is accessible from x'.

A valuation in $\mathfrak{F}$ is a function $V : (x, p_j) \to \{0, 1\}$.

If $V(x, p_j) = 1$ and xRy, then $V(y, p_j) = 1$.

$V(x, \alpha \wedge \beta) = 1$ iff $V(x, \alpha) = 1$ and $V(x, \beta) = 1$.

$V(x, \alpha \vee \beta) = 1$ iff $V(x, \alpha) = 1$ or $V(x, \beta) = 1$.

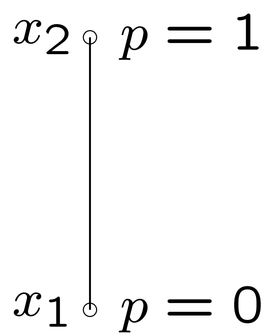
$V(x, \alpha \rightarrow \beta) = 1$ iff for all y such that xRy

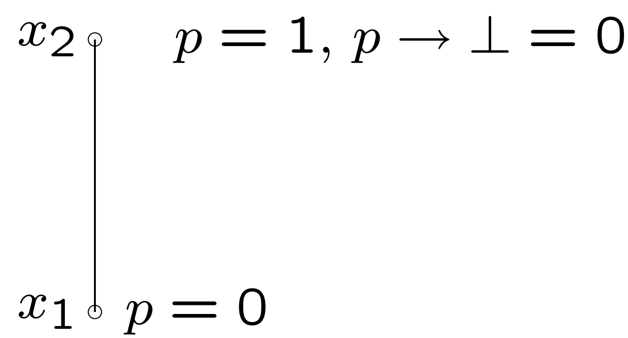
$V(y, \alpha) = 1$ implies $V(y, \beta) = 1$.

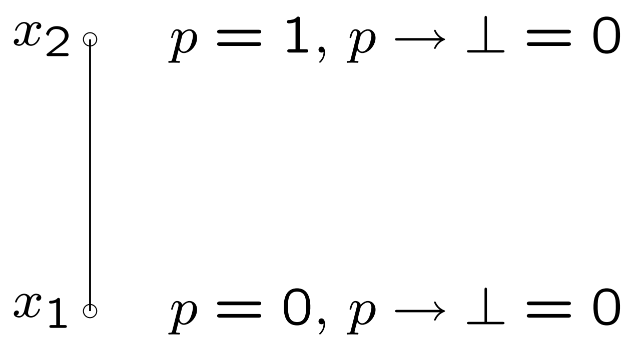
$V(x, \bot) = 0$.

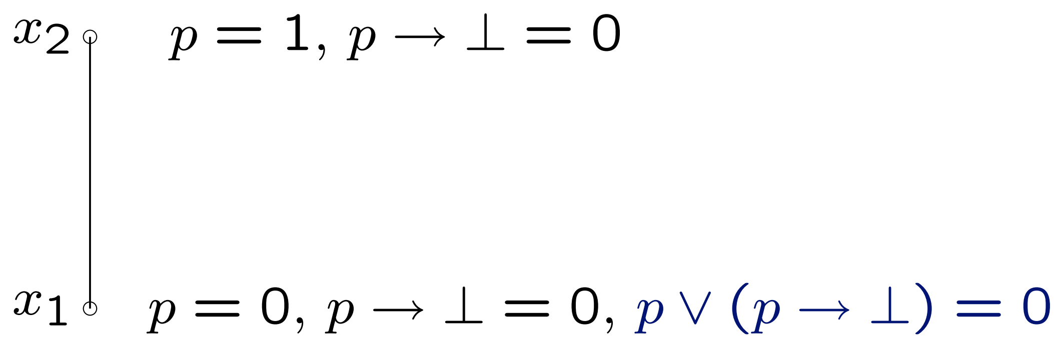
Example 1

$p \vee \neg p \notin Taut_{Int}$ where $\neg p := p \rightarrow \bot$



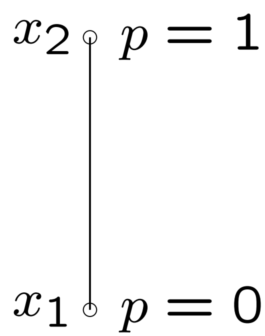

x_2
$p = 1,\ p \to \bot = 0$
x_1
$p = 0$

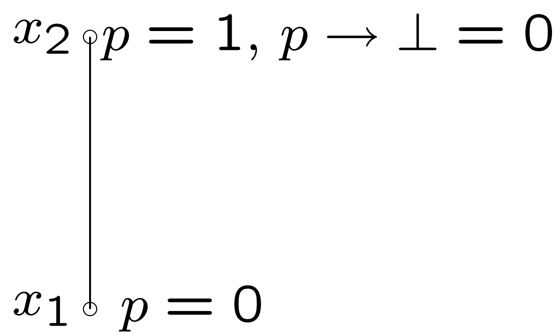

x_2
$p = 1,\ p \to \bot = 0$
x_1
$p = 0,\ p \to \bot = 0$


x_2
$p = 1$, $p \to \bot = 0$
x_1
$p = 0$, $p \to \bot = 0$, $p \vee (p \to \bot) = 0$

Example 2

$\neg\neg p \to p \notin Taut_{Int}$ where $\neg p := p \to \bot$




x_2 $p = 1$, $p \to \bot = 0$
x_1 $p = 0$

$p = 1$, $p \to \bot = 0$

$p = 0$, $p \to \bot = 0$

$p = 1$, $p \to \bot = 0$

$p = 0$, $p \to \bot = 0$

$(p \to \bot) \to \bot = 1$

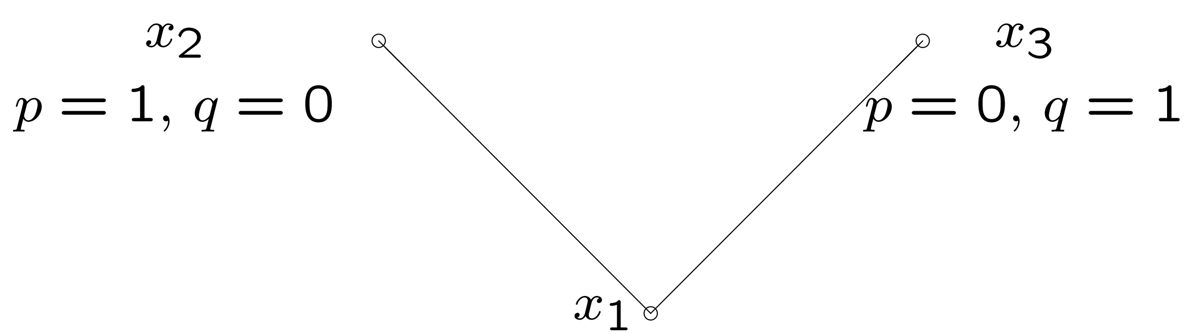
$p = 1$, $p \to \bot = 0$

$p = 0$, $p \to \bot = 0$

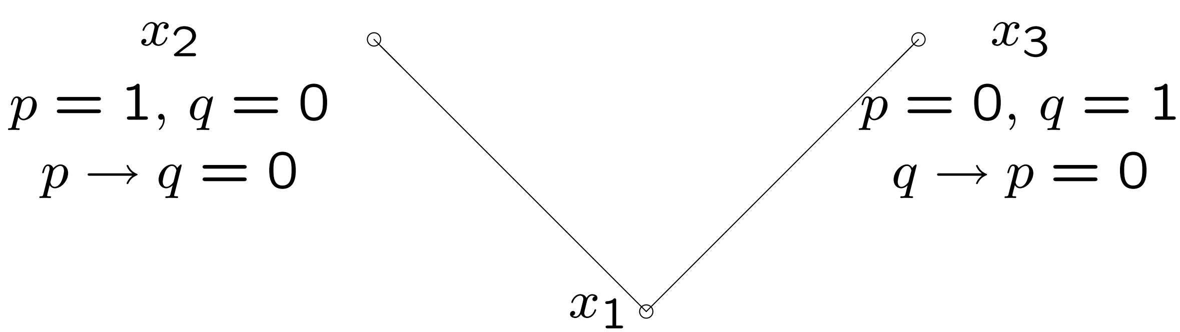
$((p \to \bot) \to \bot) \to p = 0$

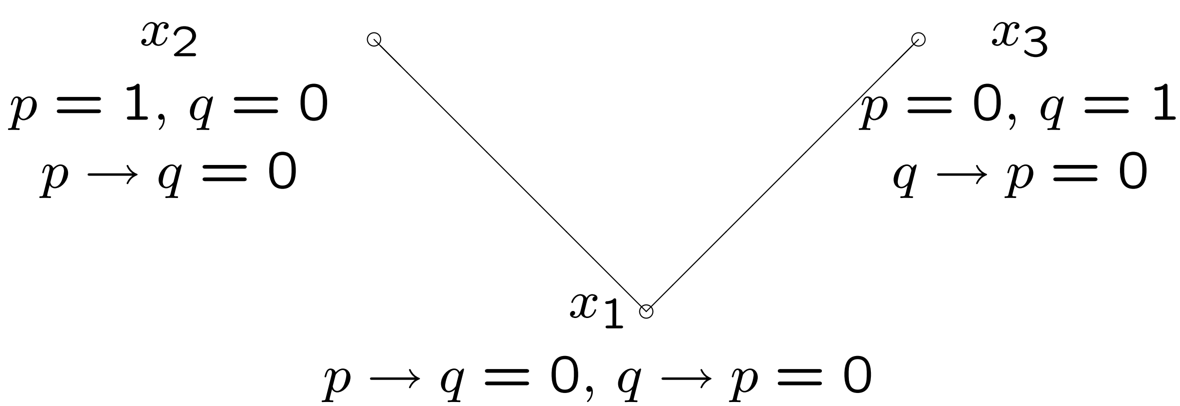
Example 3

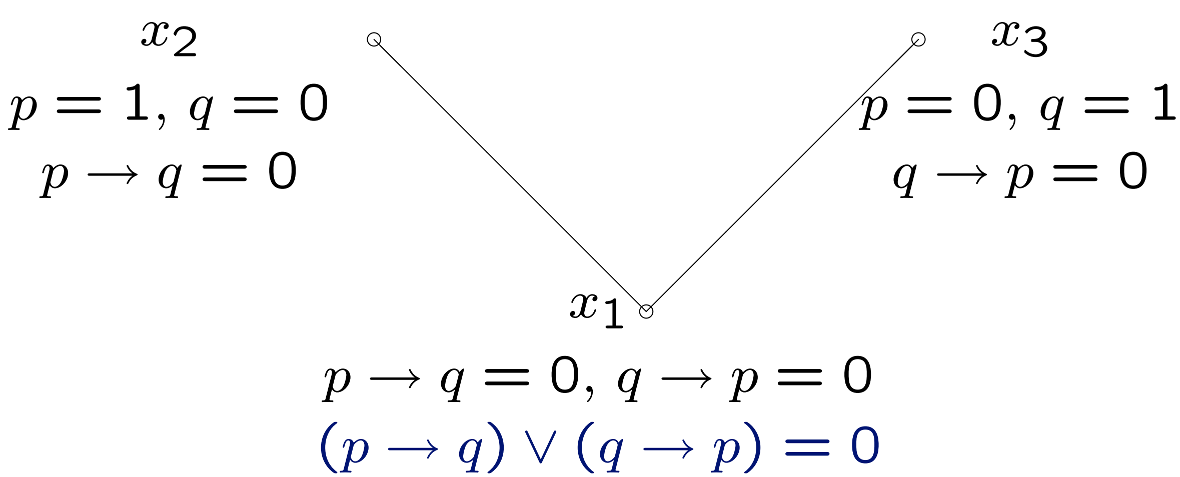
$(p \rightarrow q) \vee (q \rightarrow p) \notin Taut_{Int}$



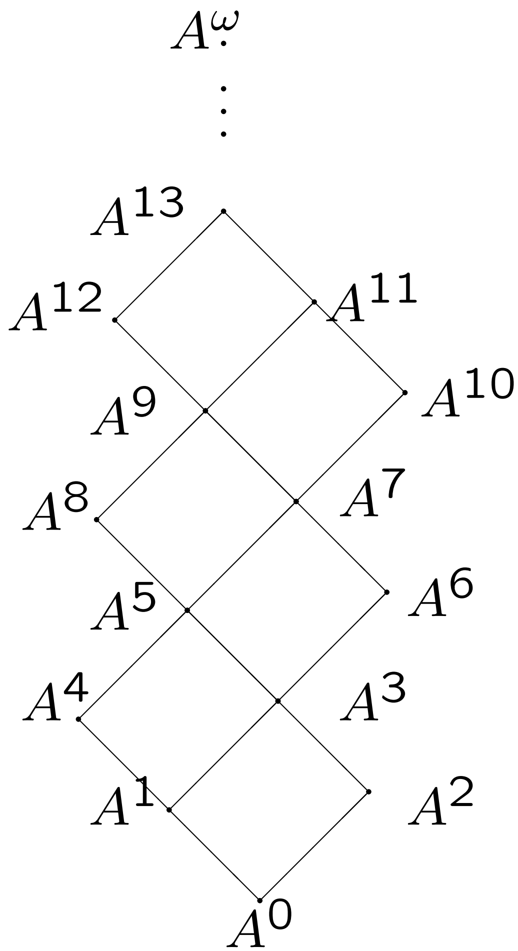
$x_1 R x_2$ and $x_1 R x_3$


x_2
$p = 1$, $q = 0$
$p \to q = 0$
x_3
$p = 0$, $q = 1$
$q \to p = 0$
x_1


x_2
$p = 1,\ q = 0$
$p \to q = 0$
x_3
$p = 0,\ q = 1$
$q \to p = 0$
x_1
$p \to q = 0,\ q \to p = 0$


x_2
$p = 1,\ q = 0$
$p \to q = 0$
x_3
$p = 0,\ q = 1$
$q \to p = 0$
x_1
$p \to q = 0,\ q \to p = 0$
$(p \to q) \vee (q \to p) = 0$

Rieger - Nishimura lattice



$$[\alpha^k]_\equiv = A^k$$

We associate the density $\mu(A)$ with a subset A of formulas as:

$$\mu(A) = \lim_{n\to\infty} \frac{\#\{t \in A : \|t\| = n\}}{\#\{t \in Form : \|t\| = n\}} \tag{1}$$

if the appropriate limit exists.

Asymptotic density is finitely additive:

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

for $A \cap B = \emptyset$.

Asymptotic density is not countably additive:

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) \neq \sum_{i=0}^{\infty} \mu\left(A_i\right)$$

But:

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) \geq \sum_{i=0}^{\infty} \mu\left(A_i\right)$$

The case of the Rieger - Nishimura lattice

Because: $\bigcup_{i=0}^{\infty} A^i \cup A^\omega = Form$ then

$$1 = \mu\left(\bigcup_{i=0}^{\infty} A^i \cup A^\omega\right) \geq \sum_{i=0}^{\infty} \mu\left(A^i\right) + \mu(A^\omega).$$

Hence if the densities exist, then

$$\sum_{i=0}^{\infty} \mu\left(A^i\right) \leq 1 \quad \text{and} \quad \mu(A^\omega) \leq 1.$$

If $\mu(A^\omega)$ exists?

Problem 5. *Does the density of truth of* $Int_1^{\to,\vee,\bot}$ *(or* $Int_1^{\to,\vee,\neg}$*) as a limit, exist?*

Problem 6. *Does the density of truth of* $Int_k^{\to,\vee,\bot}$ *(or* $Int_k^{\to,\vee,\neg}$*) as a limit, exist?*

The Drmota-Lalley-Woods theorem

Theorem 7. *Consider a nonlinear polynomial system, defined by a set of equations*

$$\{y = \Phi_j(z, y_1, ..., y_m)\}, \quad 1 \leq j \leq m$$

which is a-proper, a-positive, a-irreducible and a-aperiodic. Then

1. *All component solutions* y_i *have the same radius of convergence* $\rho < \infty$.

2. *There exist functions* h_j *analytic at the origin such that*

$$y_j = h_j(\sqrt{1 - z/\rho}), \quad (z \to \rho^-). \tag{2}$$

3. *All other dominant singularities are of the form* $\rho\omega$ *with* ω *being a root of unity.*

4. *If the system is a-aperiodic then all* y_j *have* ρ *as unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form:*

$$[z^n]y_j(z) \sim \rho^{-n} \left(\sum_{k \geq 1} d_k n^{-1-k/2} \right). \tag{3}$$

Application of the Drmota-Lalley-Woods theorem

Suppose we have two functions f_T and f_F enumerating the tautologies of some logic and all formulas. Suppose they have the same dominant singularity ρ and there are the suitable constants α_1, α_2, β_1, β_2 such that:

$$f_T(z) = \alpha_1 - \beta_1\sqrt{1 - z/\rho} + O(1 - z/\rho), \tag{4}$$

$$f_F(z) = \alpha_2 - \beta_2\sqrt{1 - z/\rho} + O(1 - z/\rho). \tag{5}$$

Then the *density of truth* (probability that a random formula is a tautology) is given by:

$$\mu(T) = \lim_{n\to\infty} \frac{[z^n]f_T(z)}{[z^n]f_F(z)} = \frac{\beta_1}{\beta_2}. \tag{6}$$

The main generating function

Language: $p, \to, \vee, \bot$.

Lemma 8. *The generating function f for the numbers $|F_n|$ is the following:*

$$f(z) = \frac{1 - \sqrt{1 - 16z}}{4}.$$

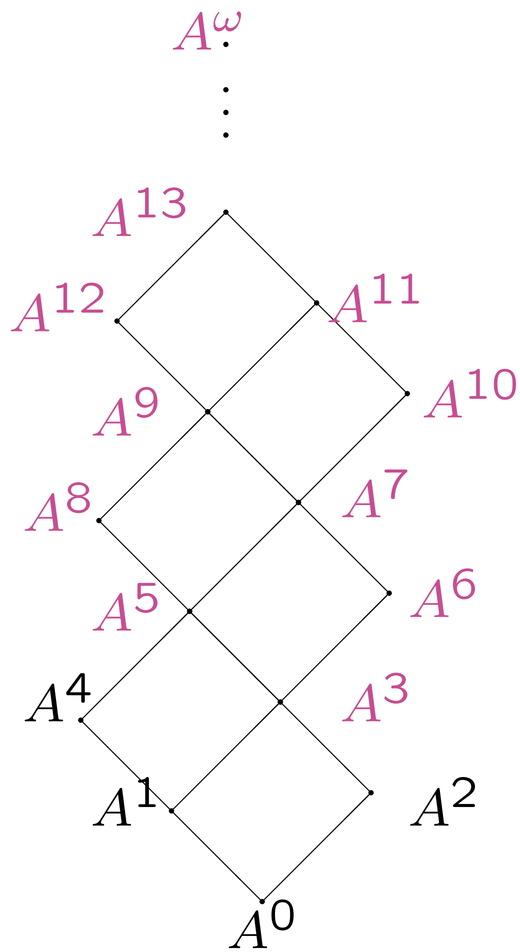
Finite quotient sub-lattices obtained from the Rieger - Nishimura lattice $\mathcal{R}$

Definition 9. *Let* $(\mathcal{B}, \leq)$ *be a pseudo-Boolean algebra (PBA). A nonempty set* $\mathcal{D} \subset \mathcal{B}$ *is a filter if for any* $a, b \in \mathcal{D}$ *it holds: 1)* $a \wedge b \in \mathcal{D}$*, 2) if* $a \in \mathcal{D}$ *and* $a \leq c$*, then* $c \in \mathcal{D}$ *.*

$[A^{2n-1}) = \{\alpha \in Form : \ \alpha^{2n-1} \rightarrow \alpha \in A^{\omega}\}$ -generated filter

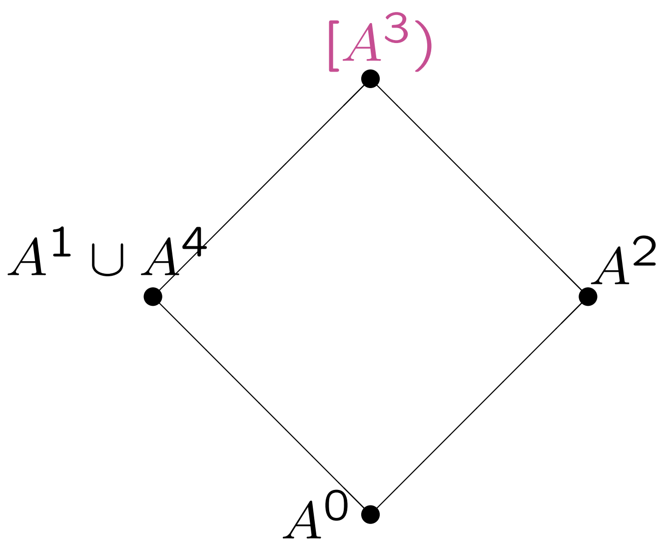
Sequence of finite quotient algebras

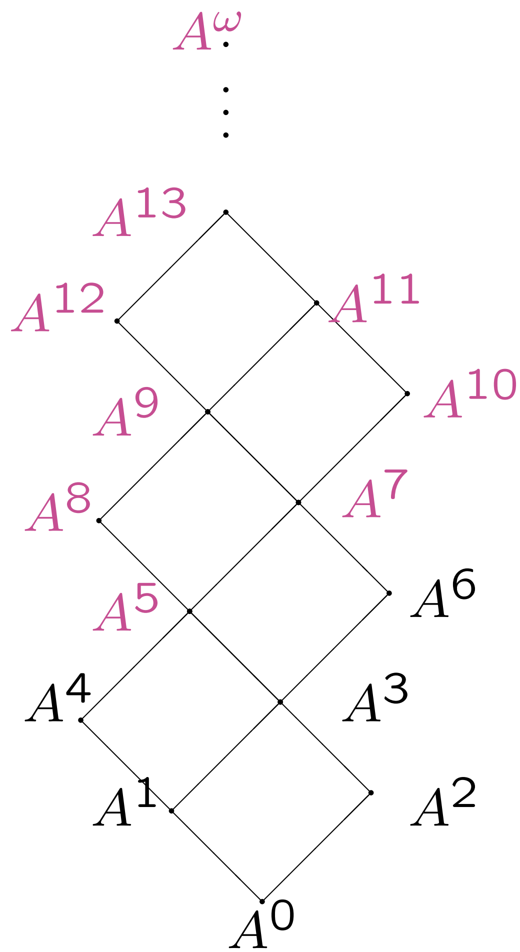
$$AL_4 := \mathcal{R}/_{[A^3)}, \quad AL_6 := \mathcal{R}/_{[A^5)}, \quad AL_8 := \mathcal{R}/_{[A^7)}, \ \ldots$$
$$AL_{2n} := \mathcal{R}/_{[A^{2n-1})}, \ldots$$



A^{ω}
A^{13}
A^{12}
A^{11}
A^{10}
A^{9}
A^{8}
A^{7}
A^{6}
A^{5}
A^{4}
A^{3}
A^{1}
A^{2}
A^{0}

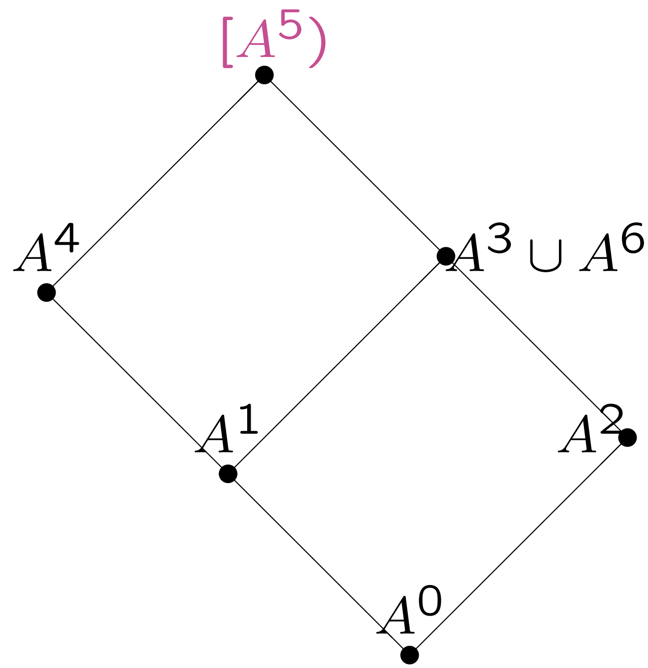
Algebra $AL_4 := \mathcal{R}/_{[A_3)}$



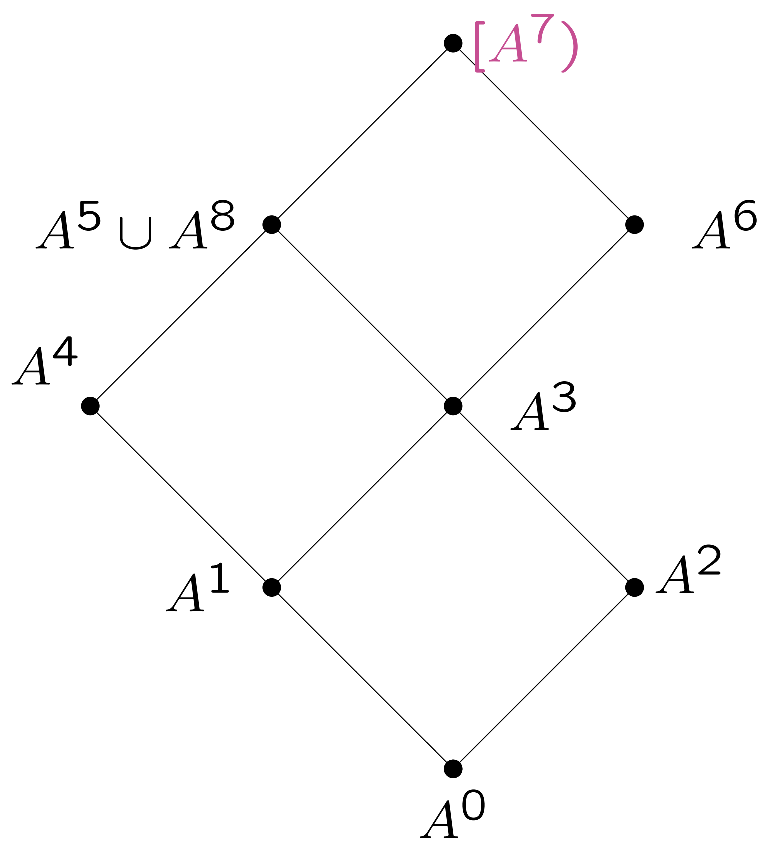


A^ω
A^{13}
A^{12}
A^{11}
A^{10}
A^9
A^8
A^7
A^6
A^5
A^4
A^3
A^1
A^2
A^0

Algebra $AL_6 := \mathcal{R}/_{[A_5)}$



Algebra $AL_8 := \mathcal{R}/_{[A_7)}$



Decreasing sequence of filters:

$$[A^3) \supset [A^5) \supset ... \supset [A^{2n-1}) \supset .... \supset A^\omega$$

Theorem 10. *The density* $\mu(A^k)$ *exists for any* $k \in N$*.*
Corollary 11. *The densities* $\mu([A^{2n-1}))$ *exist for any* $n \in \mathbb{N}$*.*

Proof. Density of classes from AL_4.

The operations $\{\to, \vee\}$ in the algebra are given by the following truth-tables:

$\to$	A^0	$A^1 \cup A^4$	A^2	$[A^3)$
A^0	$[A^3)$	$[A^3)$	$[A^3)$	$[A^3)$
$A^1 \cup A^4$	A^2	$[A^3)$	A^2	$[A^3)$
A^2	$A^1 \cup A^4$	$A^1 \cup A^4$	$[A^3)$	$[A^3)$
$[A^3)$	A^0	$A^1 \cup A^4$	A^2	$[A^3)$

$\vee$	A^0	$A^1 \cup A^4$	A^2	$[A^3)$
A^0	A^0	$A^1 \cup A^4$	A^2	$[A^3)$
$A^1 \cup A^4$	$A^1 \cup A^4$	$A^1 \cup A^4$	$[A^3)$	$[A^3)$
A^2	A^2	$[A^3)$	A^2	$[A^3)$
$[A^3)$	$[A^3)$	$[A^3)$	$[A^3)$	$[A^3)$

$$
\begin{cases}
f_0(z) = & f_{[3)}(z) f_0(z) + [f_0(z)]^2 + z \\
(f_1 + f_4)(z) = & f_{[3)}(z)(f_1 + f_4)(z) + \\
& f_2(z)[f_0(z) + (f_1 + f_4)(z)] + 2 f_0(z)(f_1 + f_4)(z) \\
& + [(f_1 + f_4)(z)]^2 + z \\
f_2(z) = & f_{[3)}(z) f_2(z) + (f_1 + f_4)(z)[f_0(z) + \\
& f_2(z)] + 2 f_0(z) f_2(z) + [f_2(z)]^2 \\
f_{[3)}(z) = & f(z) - [f_0(z) + (f_1 + f_4)(z) + f_2(z)]
\end{cases}
$$

The system is a-proper, a-positive *, a-irreducible and a-aperiodic. All the functions have the same as the function f unique dominant singularity $z_0 = 1/16$ and the densities of the classes A^0, $A^1 \cup A^4$, A^2, $[A^3)$ exist.

*For the function $f_{[3)}$ there is a strictly positive formula built from the other functions. We use the another one for simplicity

Analogous situation holds for each algebra AL_{2n} for any $n \in \mathbb{N}$. In any case we obtain a system of $2n$ equations, which is a-proper, a-positive, a-irreducible and a-aperiodic. So, the densities again exist. $\square$

Calculation of the basic functions

From system of four equations we calculate:

$$
\begin{aligned}
f_0 &= \frac{1}{4}\left(1+3f_0^* - f - \sqrt{(1+3f_0^* - f)^2 - 8z}\right) \\
f_1 + f_4 &= 2f_0^* - f^0 \\
f_2 &= f_0^* - f_0 \\
f_{[3)} &= f - f_0 - (f_1 + f_4) - f_2.
\end{aligned}
$$

where $f = \frac{1-\sqrt{1-16z}}{4}$ and $f_0^* = \frac{z}{1-f}$

Lemma 12. *Expansions of functions* f, f_0, $f_1 + f_4$, f_2 *and* $f_{[3)}$ *in a neighborhood of* $z_0 = 1/16$ *are as follows:*

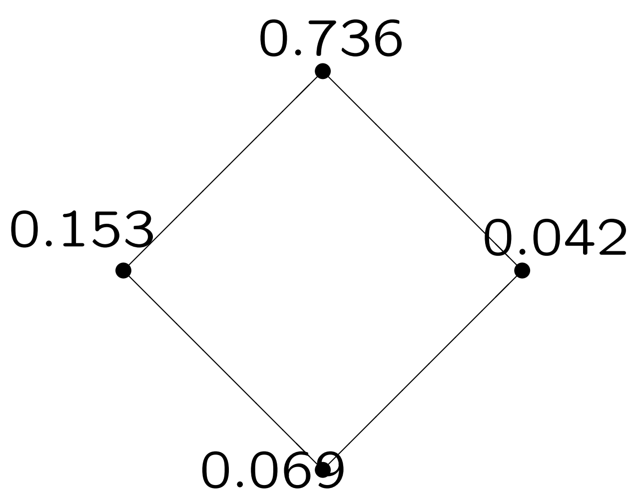
$$
\begin{aligned}
f(z) &= \frac{1}{4} - \frac{1}{4}\sqrt{1-16z} + ... \\
f_0(z) &= a_0 + a_1\sqrt{1-16z} + ... \\
(f_1 + f_4)(z) &= b_0 + b_1\sqrt{1-16z} + ... \\
f_2(z) &= c_0 + c_1\sqrt{1-16z} + ... \\
f_{[3)}(z) &= d_0 + d_1\sqrt{1-16z} + ...
\end{aligned}
$$

$a_0 \approx 0.0732\ldots,\quad a_1 \approx -0.0172\ldots,\quad b_0 \approx 0.0934\ldots,\quad b_1 \approx -0.038$

$c_0 \approx 0.0101\ldots,\quad c_1 \approx -0.0105\ldots,\quad d_0 \approx 0.0733\ldots,\quad d_1 \approx -0.184$

Lemma 13. *The densities of the classes of formulas from the algebra AL_4 exist and are the following:*

$$\mu(A^0) \approx 0.069$$
$$\mu(A^1 \cup A^4) \approx 0.153$$
$$\mu(A^2) \approx 0.042$$
$$\mu([A^3)) \approx 0.736$$

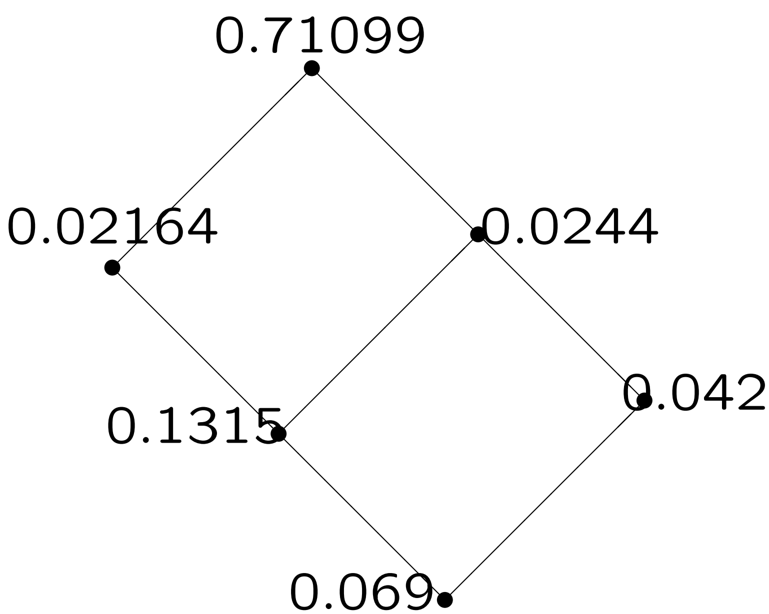


Observation 14. *The algebra AL_4 is a Lindenbaum algebra of the classical logic with one variable. Hence*

$$\mu(Cl_p^{\to,\vee,\bot}) = \mu([A^3)) \approx 0.736$$

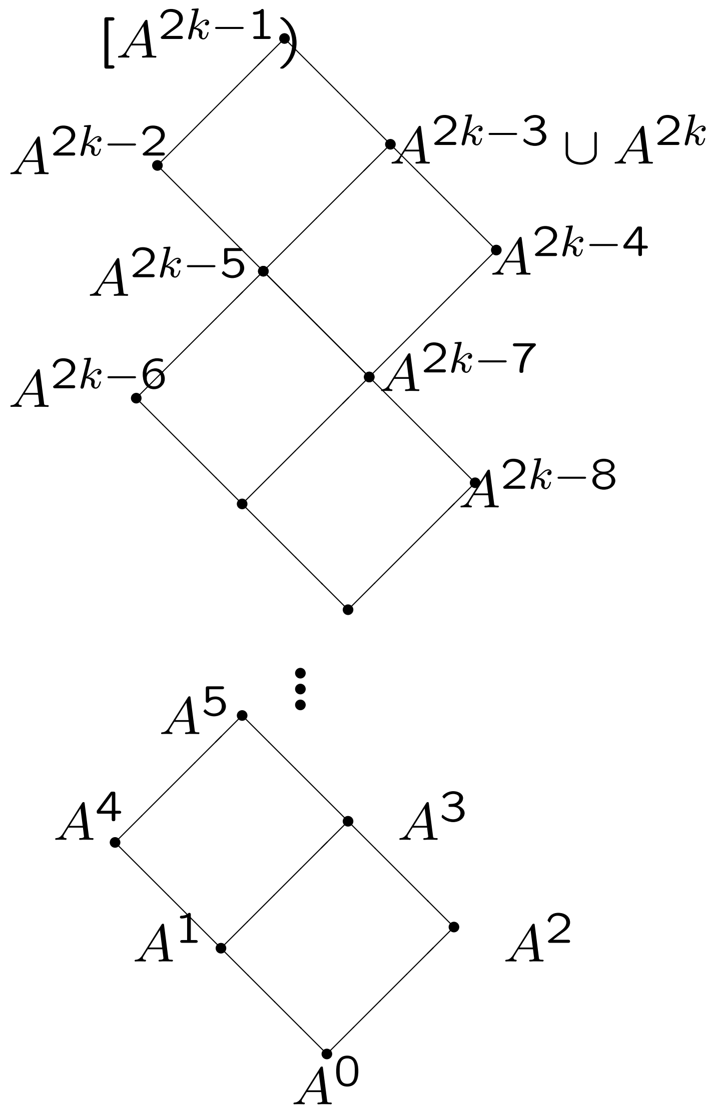
Densities of classes from AL_6

Lemma 15. *The densities of the classes from the algebra AL_6 exist and are the following:*



An upper estimation of density of $Int_p^{\rightarrow,\vee,\bot}$

We consider the algebra $AL_{2k} = \mathcal{R}/_{[A_{2n-1})}$.



[A^{2k-1})
A^{2k-2}
A^{2k-3} ∪ A^{2k}
A^{2k-5}
A^{2k-4}
A^{2k-6}
A^{2k-7}
A^{2k-8}
A^5
A^4
A^3
A^1
A^2
A^0

$$|[A^{2k-1})_n| = \sum_{i=1}^{n-1} |A_i^0| \cdot |F_{n-i}| + \sum_{i=1}^{n-1} |A_i^1| \cdot (|F_{n-i}| - |A_{n-i}^0| - |A_{n-i}^2|$$

$$+ \sum_{i=1}^{n-1} |A_i^2| \cdot (|F_{n-i}| - |A_{n-i}^0| - |A_{n-i}^1| - |A_{n-i}^4|) + ...$$

$$... + \sum_{i=1}^{n-1} (|A_i^{2k-3}| + |A_i^{2k}|) \cdot (|A_{n-i}^{2k-3}| + |A_{n-i}^{2k}| + |[A^{2k-1})_{n-i}|) +$$

$$+ \sum_{i=1}^{n-1} |A_i^{2k-2}| \cdot (|A_{n-i}^{2k-2}| + |[A^{2k-1})_{n-i}|) +$$

$$+2 \sum_{i=1}^{n-1} |A_i^{2k-2}| \cdot (|A_{n-i}^{2k-4}| + |A_{n-i}^{2k-3}| + |A_{n-i}^{2k}|)$$

$$+2 \sum_{i=1}^{n-1} |[A^{2k-1})_i| \cdot |[A^{2k-1})_{n-i}| +$$

$$+2 \sum_{i=1}^{n-1} |[A^{2k-1})_i| \cdot (|F_{n-i}| - |[A^{2k-1})_{n-i}|)$$

Of course, the sum written above as '...' is finite in the case of finite algebra AL_{2k}. After simplification, the above formula may by transformed into the following one:

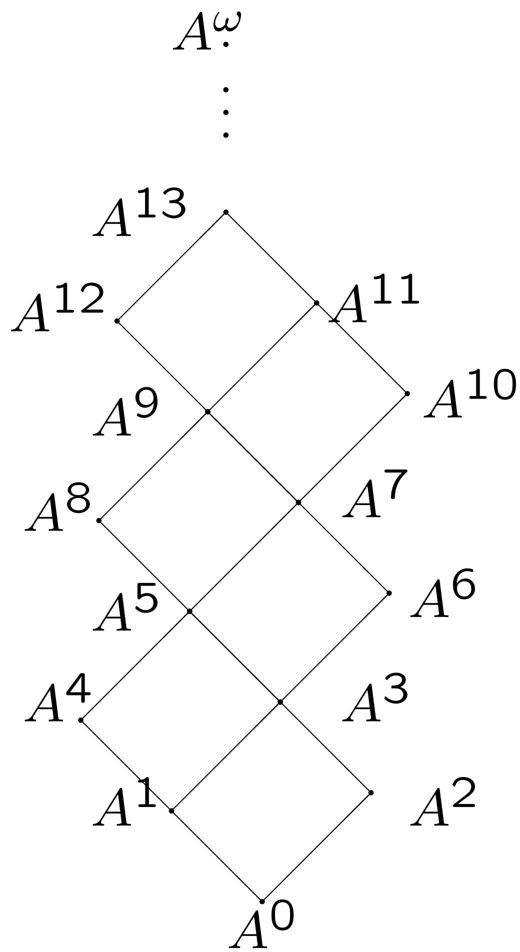
$$
\begin{aligned}
f_{[2k-1)} \;=\; & [f_0 \cdot f + f_1 \cdot (f - f_0 - f_2) + f_2 \cdot (f - f_0 - f_1 - f_4) + \\
& + f_3 \cdot (f - f_0 - f_1 - f_2 - f_4) + \dots \\
& \dots + f_{2k-2} \cdot (f - f_0 - f_1 - \dots - f_{2k-3} - f_{2k}) + (f_{2k-3} + \\
& \cdot (f - f_0 - f_1 - \dots - f_{2k-4} - f_{2k-2}) + \\
& + 2f_{2k-2} \cdot (f_{2k-4} + f_{2k-3} + f_{2k})]/(1 - 2f)
\end{aligned}
$$

$$\begin{aligned}
\mu([A_3)) &\approx 0.736 \\
\mu([A^5)) &\approx 0.71099 \\
\mu([A^7)) &\approx 0.709016 \\
\mu([A^9)) &\approx 0.709011 \\
&\vdots
\end{aligned}$$

$$\mu([A^3)) \geq \mu([A^5)) \geq \ldots \geq \mu([A^{2n-1})) \geq \ldots \geq \mu(A^\omega)$$

Observation 16. *If $\mu(A^\omega)$ exists, then $\mu(A^\omega) < 0.709011$*

A lower estimation of $\mu(A^\omega)$.


A^ω
A^{13}
A^{12}
A^{11}
A^{10}
A^9
A^8
A^7
A^6
A^5
A^4
A^3
A^1
A^2
A^0

$$|A_n^\omega| = \sum_{i=1}^{n-1} |A_i^0| \cdot |F_{n-i}| + \sum_{i=1}^{n-1} |A_i^1| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^2|)) +$$

$$+ \sum_{i=1}^{n-1} |A_i^2| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^4|)) +$$

$$+ \sum_{i=1}^{n-1} |A_i^3| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^2| + |A_{n-i}^4|)) +$$

$$+ \sum_{i=1}^{n-1} |A_i^4| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^2| + |A_{n-i}^3| + |A_{n-i}^6|))$$

$$+ \sum_{i=1}^{n-1} |A_i^5| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + ... + |A_{n-i}^4| + |A_{n-i}^6|)) +$$

$$+ ..... +$$

$$+2 \sum_{i=1}^{n-1} |A_i^\omega| \cdot |A_{n-i}^\omega| + 2 \sum_{i=1}^{n-1} |A_i^\omega| \cdot (|F_{n-i}| - |A_{n-i}^\omega|)$$

Observation 17. *Let* (c_n), (d_n) *and* (e_n) *be three sequences of natural numbers, such that* $c_n \leq d_n$ *for all* $n \in \mathbb{N}$. *Suppose two new sequences are defined recursively as follows:*

$$x_n = c_n + \sum_{i=1}^{n-1} e_i \cdot x_{n-i}, \qquad y_n = d_n + \sum_{i=1}^{n-1} e_i \cdot y_{n-i}$$

Then $x_n \leq y_n$ *for any* $n \in \mathbb{N}$.

$$|A_n^\omega| = \sum_{i=1}^{n-1} |A_i^0| \cdot |F_{n-i}| + \sum_{i=1}^{n-1} |A_i^1| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^2|))+$$

$$+ \sum_{i=1}^{n-1} |A_i^2| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^4|))+$$

$$+ \sum_{i=1}^{n-1} |A_i^3| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^2| + |A_{n-i}^4|))+$$

$$+ \sum_{i=1}^{n-1} |A_i^4| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^2| + |A_{n-i}^3| + |A_{n-i}^6|))$$

$$+ \sum_{i=1}^{n-1} |A_i^5| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + \ldots + |A_{n-i}^4| + |A_{n-i}^6|))+$$

$$+ \ldots.. +$$

$$+2 \sum_{i=1}^{n-1} |A_i^\omega| \cdot |A_{n-i}^\omega| + 2 \sum_{i=1}^{n-1} |A_i^\omega| \cdot (|F_{n-i}| - |A_{n-i}^\omega|)$$

Smaller numbers $|B_n^5|$:

$$
\begin{aligned}
|B_n^5| = & \sum_{i=1}^{n-1} |A_i^0| \cdot |F_{n-i}| + \sum_{i=1}^{n-1} |A_i^1| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^2|)) + \\
& + \sum_{i=1}^{n-1} |A_i^2| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^4|)) + \\
& + \sum_{i=1}^{n-1} |A_i^3| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^2| + |A_{n-i}^4|)) + \\
& + \sum_{i=1}^{n-1} |A_i^4| \cdot (|F_{n-i}| - (|A_{n-i}^0| + |A_{n-i}^1| + |A_{n-i}^2| + |A_{n-i}^3| + |A_{n-i}^6|)) \\
& + \sum_{i=1}^{n-1} |A_i^5| \cdot (|F_{n-i}| - (|A_{n-i}^0| + ... + |A_{n-i}^4| + |A_{n-i}^6|)) + \\
& + 2 \sum_{i=1}^{n-1} |B_i^5| \cdot |B_{n-i}^5| + 2 \sum_{i=1}^{n-1} |B_i^5| \cdot (|F_{n-i}| - |B_{n-i}^5|)
\end{aligned}
$$

$$
\begin{aligned}
g_5 \;=\; & (f_0 \cdot f + f_1 \cdot (f - f_0 - f_2) + f_2 \cdot (f - f_0 - f_1 - f_4))+ \qquad (7\\
& f_3 \cdot (f - f_0 - f_1 - f_2 - f_4) + f_4 \cdot (f - f_0 - f_1 - f_2 - f_3 - f_6)\\
& + f_5 \cdot (f - f_0 - f_1 - f_2 - f_3 - f_4 - f_6))\,/(1 - 2f).
\end{aligned}
$$

Lemma 18. *The density of the class B^5 exists and is the following:*

$$
\mu(B^5) \approx 0.7068 \tag{8}
$$

Theorem 19. *If the density of the class A^ω exists, then it is estimated as follows:*

$$0.7068 \leq \mu(A^\omega) \leq 0.709011$$

The existence of $\mu(A^\omega)$.

$$|B_n^{2k}| = \sum_{i=1}^{n-1} |A_i^0| \cdot |F_{n-i}| + \sum_{i=1}^{n-1} |A_i^1| \cdot (|F_{n-i}| - |A_{n-i}^0| - |A_{n-i}^2|) +$$

$$+ \sum_{i=1}^{n-1} |A_i^2| \cdot (|F_{n-i}| - |A_{n-i}^0| - |A_{n-i}^1| - |A_{n-i}^4|) + \ldots$$

$$\ldots + \sum_{i=1}^{n-1} |A_i^{2k}| \cdot (|F_{n-i}| - |A_{n-i}^0| - |A_{n-i}^1| - \ldots - |A_{n-i}^{2k-1}| - |A_{n-i}^{2k+2}|)$$

$$+2 \sum_{i=1}^{n-1} |B_i^{2k}| \cdot |B_{n-i}^{2k}| + 2 \sum_{i=1}^{n-1} |B_i^{2k}| \cdot (|F_{n-i}| - |B_{n-i}^{2k}|).$$

Observation:

$$B^4 \subset B^6 \subset \ldots \subset A^\omega \subset \ldots \subset [A^5) \subset [A^3) \ .$$

Sequence of compartments $[\mu(B^{2k}), \mu([A^{2k-1}))]$, for $k \geq 2$.

Problem: to show that their 'lengths' tend to 0.

Lemma 20.

$$\lim_{k \to \infty} \left(\mu([A^{2k-1})) - \mu(B^{2k}) \right) = 0$$

Proof. We consider the numbers $|[A^{2k-1})_n| - |B_n^{2k}|$.

$$
\begin{aligned}
|[A^{2k-1})_n| - |B_n^{2k}| \;=\; & 2\sum_{i=1}^{n-1} |A_i^{2k-2}| \cdot (|A_{n-i}^{2k-4}| + |A_{n-i}^{2k-3}| + |A_{n-i}^{2k}|) + \\
& + \sum_{i=1}^{n-1} |A_i^{2k}| \cdot (|A_{n-i}^{2k-3}| + |A_{n-i}^{2k-1}| + |A_{n-i}^{2k+2}|) + \\
& + 2\sum_{i=1}^{n-1} |F_i| \cdot (|[A^{2k-1})_{n-i}| - |B_{n-i}^{2k}|) - \\
& - \sum_{i=1}^{n-1} |A_i^{2k-1}| \cdot |[A^{2k-1})_{n-i}|
\end{aligned}
$$

The numbers $\sum_{i=1}^{n-1} |A_i^{2k-1}| \cdot |[A^{2k-1})_{n-i}|$ are non-negative, so on the base of Observation we may consider larger numbers $|C_n^k|$:

$$|C_n^k| = 2\sum_{i=1}^{n-1} |A_i^{2k-2}| \cdot (|A_{n-i}^{2k-4}| + |A_{n-i}^{2k-3}| + |A_{n-i}^{2k}|) +$$
$$+ \sum_{i=1}^{n-1} |A_i^{2k}| \cdot (|A_{n-i}^{2k-3}| + |A_{n-i}^{2k-1}| + |A_{n-i}^{2k+2}|) + 2\sum_{i=1}^{n-1} |F_i| \cdot |C_n^k$$

The numbers $|C_n^k|$ characterize the set C^k consisting of formulas being disjunctions between formulas from A^{2k-2} and $A^{2k-4} \cup A^{2k-3} \cup A^{2k}$, and implications from A^{2k} to $A^{2k-3} \cup A^{2k-1} \cup A^{2k+2}$, and disjunctions between formulas from C^k and formulas from $Form$.

From the above we obtain formulas defining the generating functions f_{C^k} for the numbers $|C_n^k|$:

$$f_{C^k} \ = \ \frac{[f_{2k} \cdot (f_{2k-3} + f_{2k-1}) + 2f_{2k-2} \cdot (f_{2k-4} + f_{2k-3} + f_{2k})]}{1 - 2f}.$$

The function f_{C^k} is defined by functions with dominant singularity at $z_0 = 1/16$ (see proof of Theorem 10). So, it has the same dominant singularity. The density of the class C^k can be computed as follows:

$$\mu(C^k) \ = \ \frac{f'_{C^k}(\frac{1}{16})}{f'(\frac{1}{16})}.$$

We show that the values of $f'_{C^k}(\frac{1}{16})$ tend to 0 when k tends to infinity. For simplicity, we introduce a new symbol

$h_k := f_{2k} \cdot (f_{2k-3} + f_{2k-1}) + 2f_{2k-2} \cdot (f_{2k-4} + f_{2k-3} + f_{2k})$.
Then, from (10), we have:

$$f'_{C^k}(\frac{1}{16}) = \frac{h'_k(\frac{1}{16}) \cdot (1 - 2f(\frac{1}{16})) - h_k(\frac{1}{16}) \cdot (-2f'(\frac{1}{16}))}{(1 - 2f(\frac{1}{16}))^2}.$$

The values of $f(\frac{1}{16})$ and $f'(\frac{1}{16})$ exist and are constant. To prove that

$$\lim_{k \to \infty} h'_k \left(\frac{1}{16}\right) = 0 \quad \text{and} \quad \lim_{k \to \infty} h_k \left(\frac{1}{16}\right) = 0. \quad (10)$$

we prove that

$$\lim_{k \to \infty} f'_k \left(\frac{1}{16}\right) = 0 \quad \text{and} \quad \lim_{k \to \infty} f_k \left(\frac{1}{16}\right) = 0. \quad (11)$$

It is straightforward to observe that (11) yields (10). From Theorem 10 it follows that $\mu(A^k)$ exists for each $k \in \mathbb{N}$.

The series $\sum_{k=0}^{\infty} \mu\left(A^k\right)$ is convergent and hence $\lim_{k\to\infty} \mu\left(A^k\right) = 0$.

So, from the transfer lemma (we know that the functions f_k have the same dominant singularity) we obtain:

$$\lim_{k\to\infty} f_k'\left(\frac{1}{16}\right) = 0. \tag{12}$$

Similarly, let us consider the series $\sum_{k=0}^{\infty} f_k\left(\frac{1}{16}\right)$. This series is bounded by $f\left(\frac{1}{16}\right) = \frac{1}{2}$ and the values $f_k\left(\frac{1}{16}\right)$ are non-negative [†], so it also must be convergent. Hence:

$$\lim_{k\to\infty} f_k\left(\frac{1}{16}\right) = 0. \tag{13}$$

□

By Theorem 19 and Lemma 20 we get:

Theorem 21. *The density of the class A^ω exists and is about 70%.*

[†]It could be justified as follows: for each $i \in \mathbb{N}$, $f_i(1/16) \geq 0$ because $f_i(z) = \sum_{n=0}^{\infty} a_{in} z^n$ and the series is convergent at $z_0 = \frac{1}{16}$ and then the sum $\sum_{n=0}^{\infty} a_{in} (\frac{1}{16})^n$ is also non-negative.

Problem 1 Investigate the logics $Int_k^{\to,\vee,\bot}$ and $Cl_k^{\to,\vee,\bot}$ and give an answer if they are asymptotically identical.