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On the existence of a continuum of logics in NEXT($\mathbf{KTB}$)

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Extension of the Brouwer logic $\mathbf{KTB}$

$\mathbf{T_n} = \mathbf{KTB} \oplus (4_n)$, where

K	$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
T	$\Box p \rightarrow p$
B	$p \rightarrow \Box\Diamond p$
(4_n)	$\Box^n p \rightarrow \Box^{n+1} p$

$$(tran_n) \quad \forall_{x,y}(\text{if } xR^{n+1}y \text{ then } xR^n y)$$

where the relation of n-step accessibility is defined inductively as follows:

$$\begin{array}{rcl} xR^0y & \text{iff} & x = y \\ xR^{n+1}y & \text{iff} & \exists_z\,(xR^nz \,\wedge\, zRy) \end{array}$$

$$\mathbf{KTB} \subset ... \subset \mathbf{T_{n+1}} \subset \mathbf{T_n} \subset ... \subset \mathbf{T_2} \subset \mathbf{T_1} = \mathbf{S5}.$$

Kripke frames for $\mathbf{T_2}$ logic

A Kripke frame is a pair $\mathfrak{F} = \langle W, R \rangle$, where the relation R is reflexive, symmetric and 2-transitive.

Denote $\alpha := p \wedge \neg\Diamond\Box p$.

Definition 1.

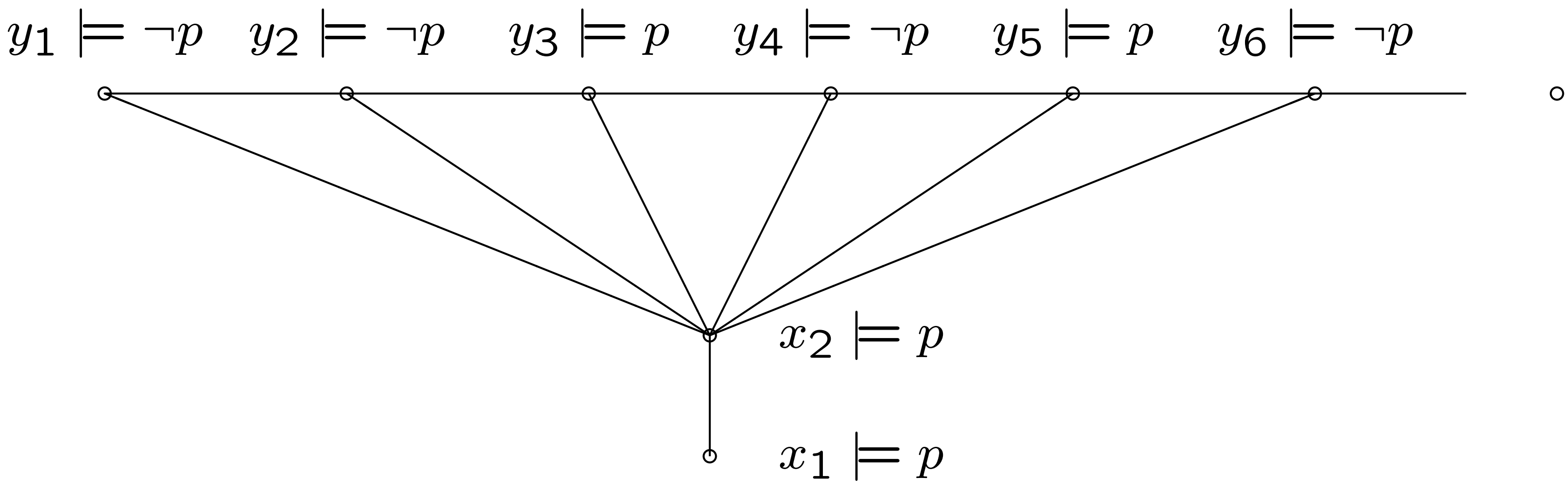
$$A_1 := \neg p \wedge \Box\neg\alpha$$
$$A_2 := \neg p \wedge \neg A_1 \wedge \Diamond A_1$$
$$A_3 := \alpha \wedge \Diamond A_2$$

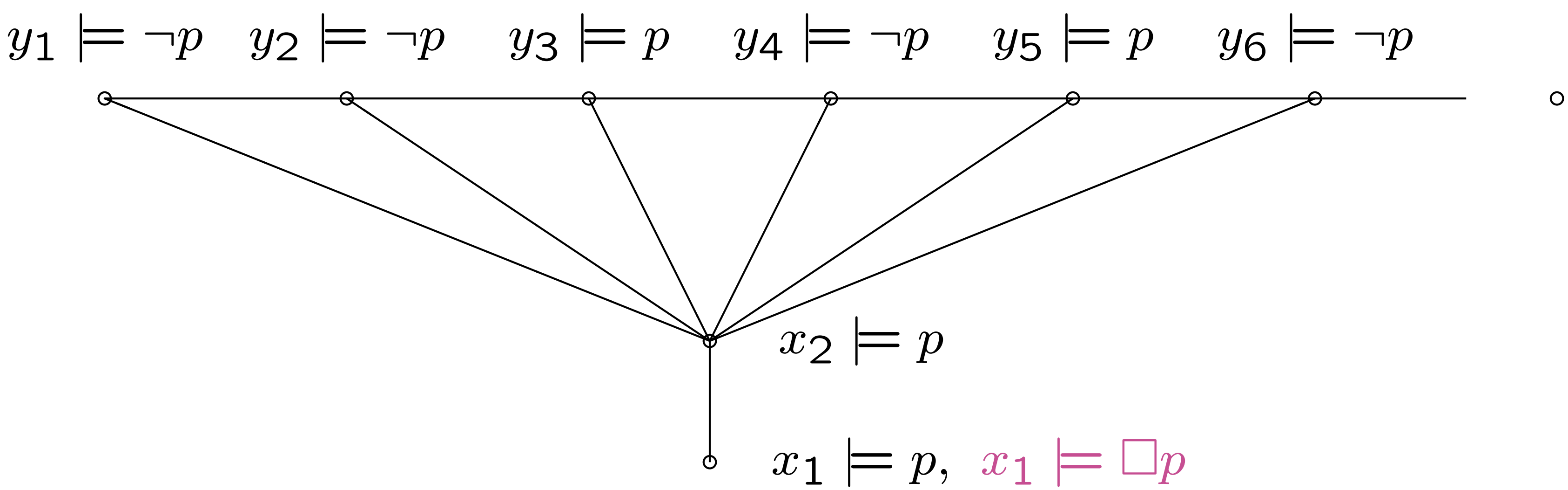
For $n \geq 2$:

$$\begin{aligned} A_{2n} &:= \neg p \wedge \Diamond A_{2n-1} \wedge \neg A_{2n-2} \\ A_{2n+1} &:= \alpha \wedge \Diamond A_{2n} \wedge \neg A_{2n-1} \end{aligned}$$

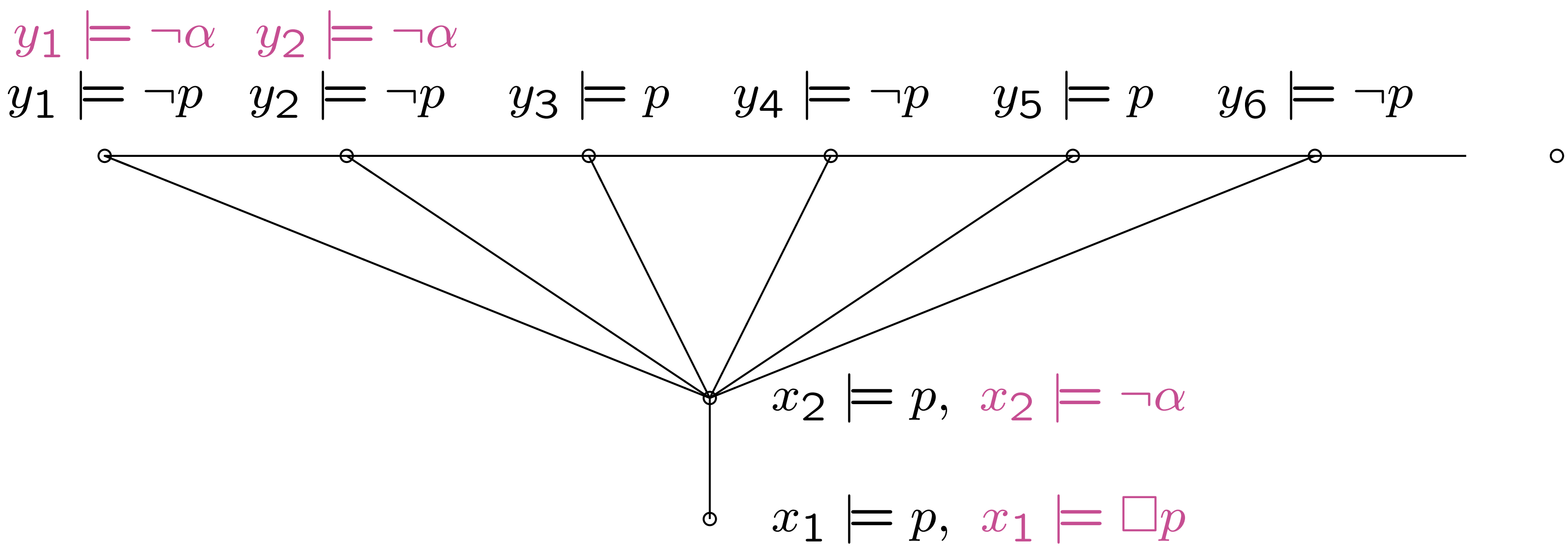
Theorem 2. *The formulas $\{A_i\}$, $i \geq 1$ are non-equivalent in the logic $\mathbf{T_2}$.*

Proof. Let us take the following model $\mathfrak{M} = \langle W, R, V \rangle$:

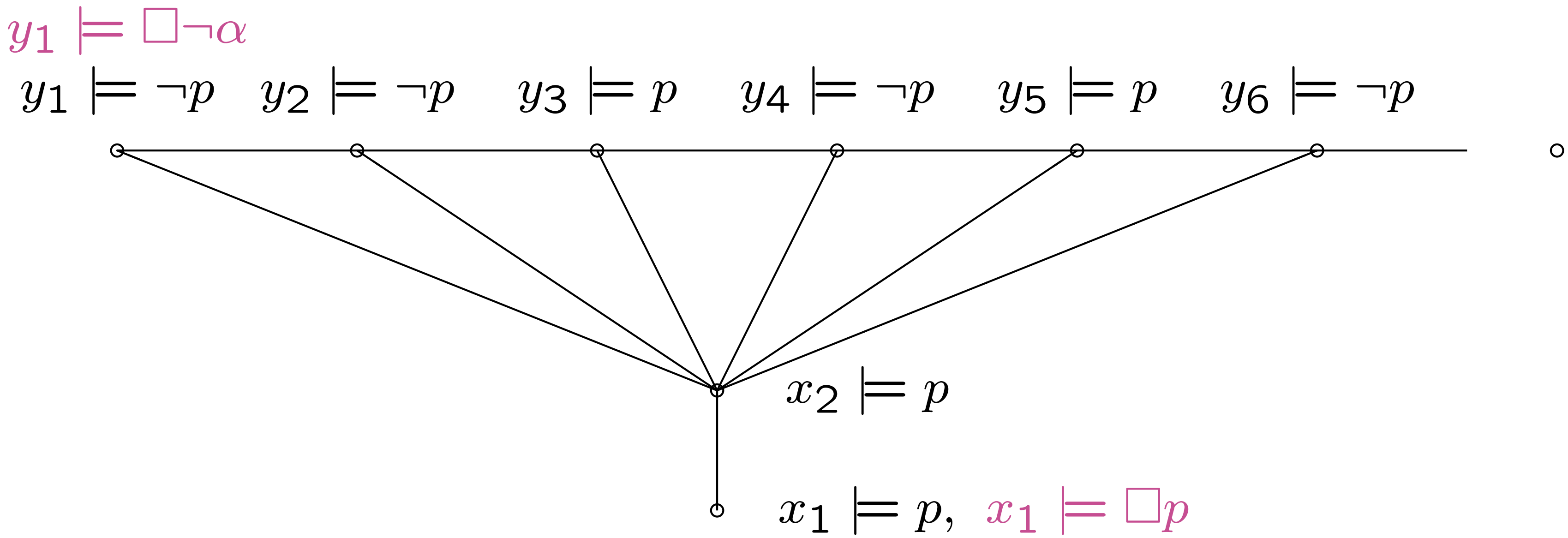

$y_1 \models \neg p$
$y_2 \models \neg p$
$y_3 \models p$
$y_4 \models \neg p$
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$



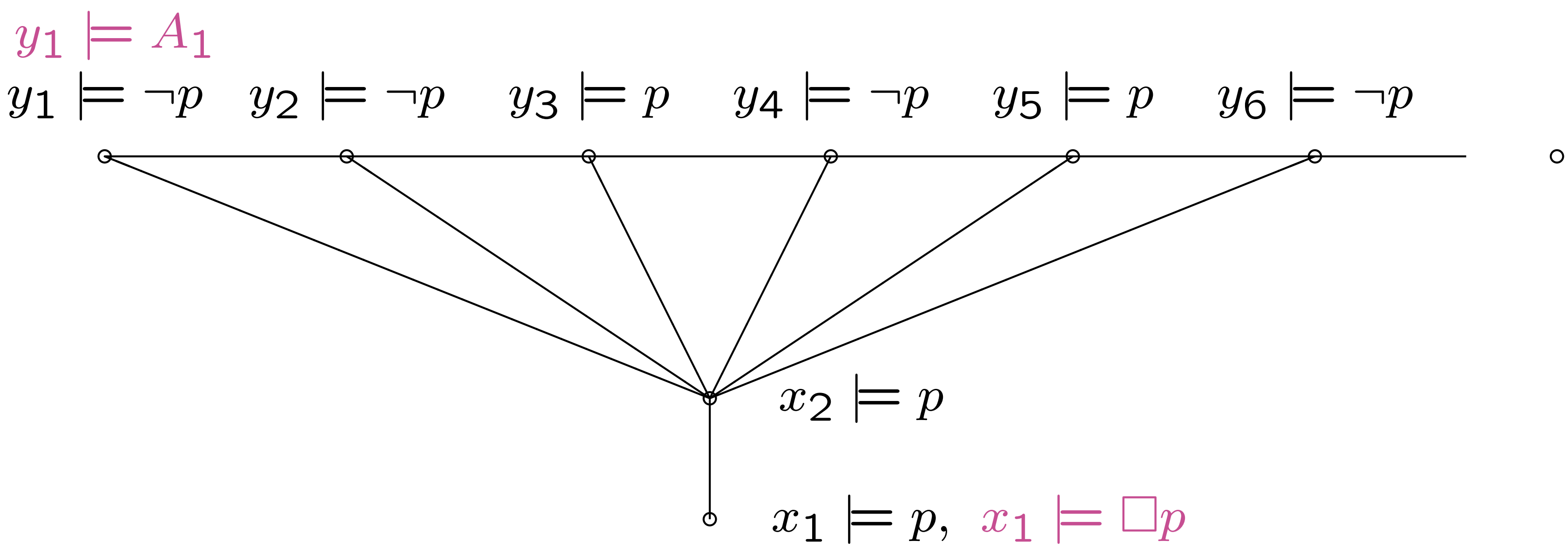
$y_1 \models \neg p$
$y_2 \models \neg p$
$y_3 \models p$
$y_4 \models \neg p$
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$, $x_1 \models \Box p$

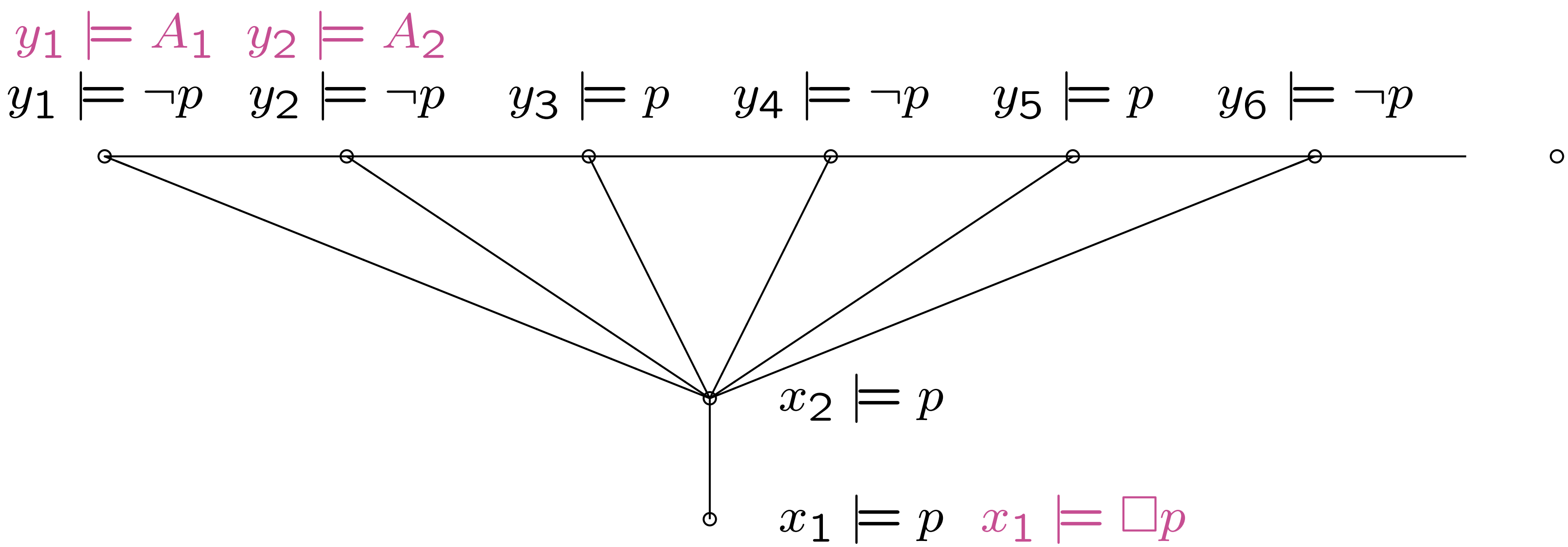


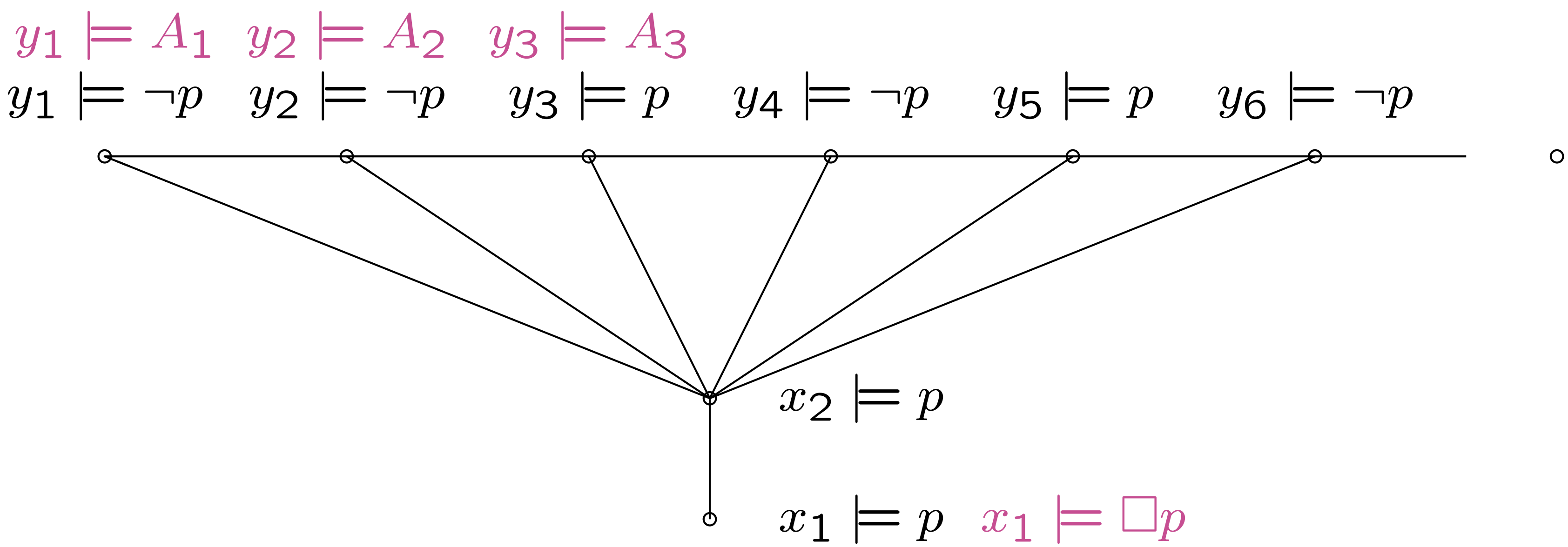
where $\alpha := p \wedge \neg\Diamond\Box p$.



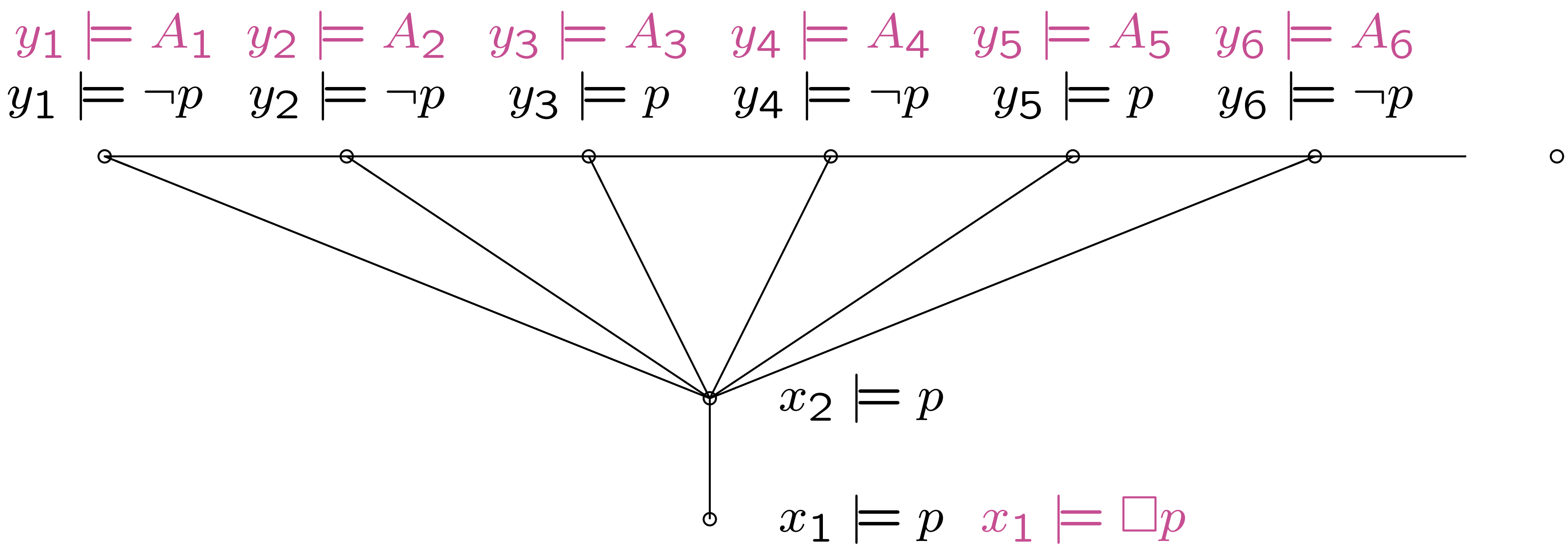
y1 ⊨ □¬α
y1 ⊨ ¬p
y2 ⊨ ¬p
y3 ⊨ p
y4 ⊨ ¬p
y5 ⊨ p
y6 ⊨ ¬p
x2 ⊨ p
x1 ⊨ p, x1 ⊨ □p


$y_1 \models A_1$
$y_1 \models \neg p$
$y_2 \models \neg p$
$y_3 \models p$
$y_4 \models \neg p$
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$, $x_1 \models \Box p$


$y_1 \models A_1$ $y_2 \models A_2$
$y_1 \models \neg p$
$y_2 \models \neg p$
$y_3 \models p$
$y_4 \models \neg p$
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$ $x_1 \models \Box p$



$y_1 \models A_1$
$y_2 \models A_2$
$y_3 \models A_3$
$y_1 \models \neg p$
$y_2 \models \neg p$
$y_3 \models p$
$y_4 \models \neg p$
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$
$x_1 \models \Box p$


$y_1 \models A_1$
$y_2 \models A_2$
$y_3 \models A_3$
$y_4 \models A_4$
$y_5 \models A_5$
$y_6 \models A_6$
$y_1 \models \neg p$
$y_2 \models \neg p$
$y_3 \models p$
$y_4 \models \neg p$
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$
$x_1 \models \Box p$

For any $i \geq 1$ and for any $x \in W$ the following holds:

$$x \models A_i \quad \text{iff} \quad x = y_i$$

Theorem 3. *There are infinitely many non-equivalent formulas written in one variable in the logic $\mathbf{T_2}$.*

[1] Kostrzycka Z., *On formulas in one variable in $NEXT(KTB)$*, Bulletin of the Section of Logic, Vol.35:2/3, (2006), 119-131.

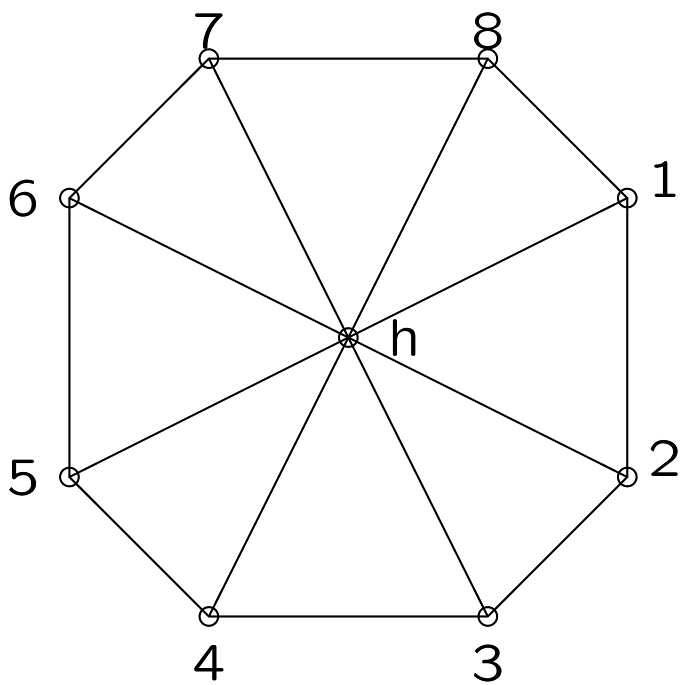
Wheel frames

Definition 4. *Let* $n \in \omega$ *and* $n \geq 5$. *The wheel frame* $\mathfrak{W}_n = \langle W, R \rangle$ *where*

$W = rim(W) \cup h$ *and* $rim(W) := \{1, 2, ..., n\}$ *and* $h \notin rim(W)$.

$R := \{(x, y) \in (rim(W))^2 : |x - y| \leq 1 (mod\,(n - 1))\} \cup \{(h, h)\} \cup \{(h, x), (x, h) : x \in rim(W)\}$.

A diagram of the $\mathfrak{W}_8$



Lemma 5. *For* $m > n \geq 5$, $L(\mathfrak{W}_n) \not\subseteq L(\mathfrak{W}_m)$.
Lemma 6. *For* $m \geq n \geq 5$, *suppose there is a p-morphism from* $\mathfrak{W}_m$ *to* $\mathfrak{W}_n$. *Then m is divisible by n.*

On the base of these two lemmas and by using the splitting technique effectively, Y. Miyazaki constructed a continuum of normal modal logics over $\mathbf{T_2}$ logic.

[2] Miyazaki Y. *Normal modal logics containing KTB with some finiteness conditions*, Advances in Modal Logic, Vol.5, (2005), 171-190.

Let:

$$\beta \;:=\; \neg\Box p \wedge \Diamond\Box p$$

$$\gamma \;:=\; \beta \wedge \Diamond A_1 \wedge \neg\Diamond A_2$$

$$\varepsilon \;:=\; \beta \wedge \neg\Diamond A_1 \wedge \neg\Diamond A_2$$

$$C_k \;:=\; \Box^2[A_{k-1} \rightarrow \Diamond A_k], \quad \text{for} \quad k > 2$$

$$D_k \;:=\; \Box^2[(A_k \wedge \neg\Diamond A_{k+1}) \rightarrow \Diamond\varepsilon],$$

$$E \;:=\; \Box^2(\Box p \rightarrow \Diamond\gamma)$$

$$F_k \;:=\; (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E) \rightarrow \Diamond^2 A_k.$$

Lemma 7. *Let* $k \geq 5$ *and* k*- odd number.*

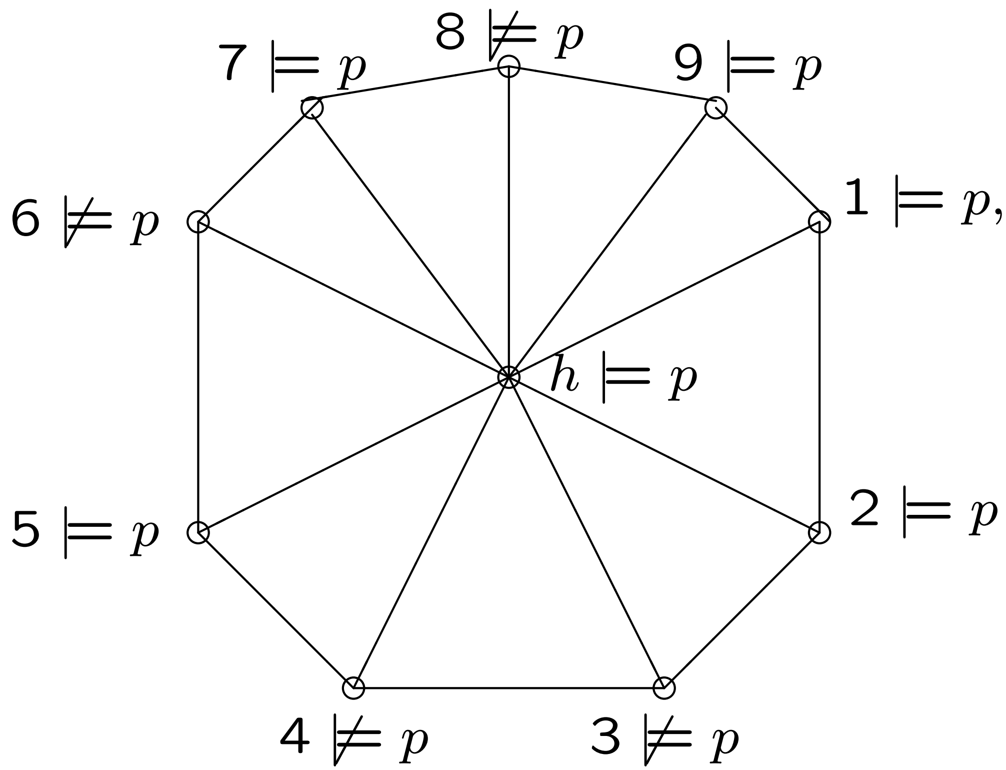
$\mathfrak{W}_i \not\models F_k$ *iff* i *is divisible by* $k+2$.

Proof. $(\Leftarrow)$

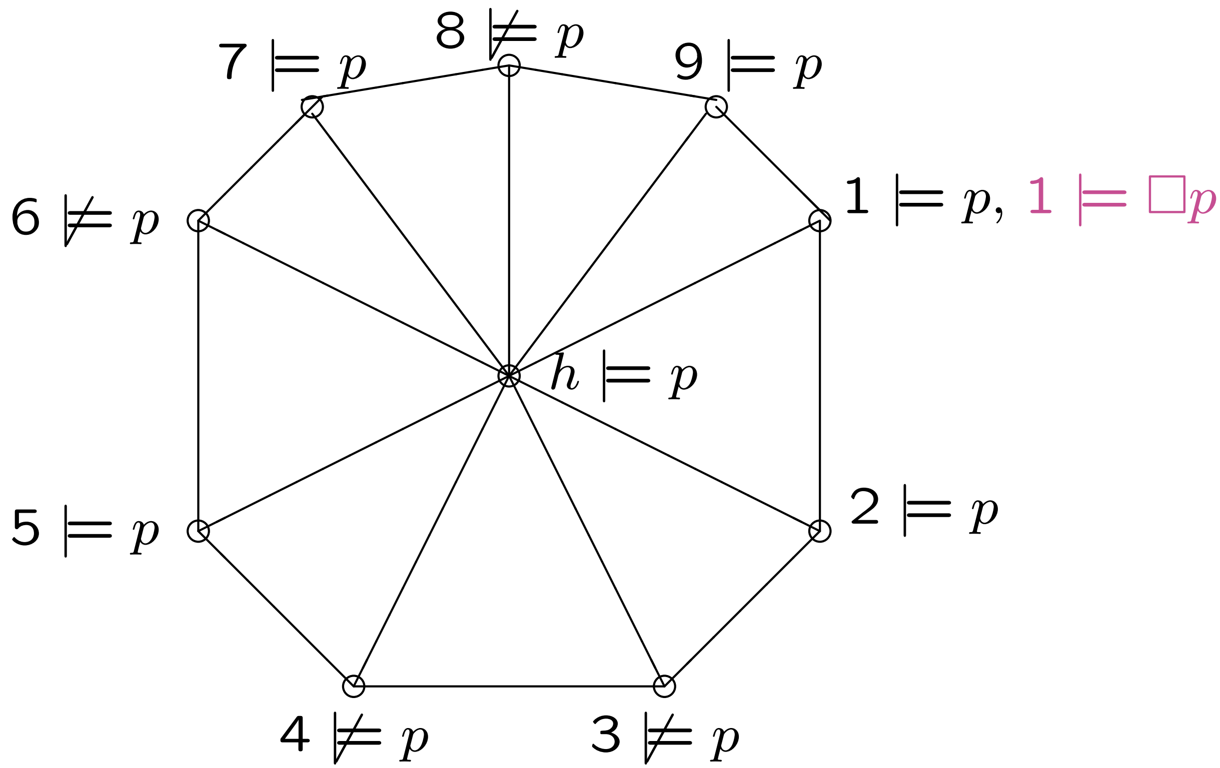
Let $i = k+2$. We define the following valuation in the frame $\mathfrak{W}_i$:

$$
\begin{array}{rcl}
h & \models & p, \\
1 & \models & p, \\
2 & \models & p, \\
3 & \not\models & p, \\
4 & \not\models & p, \\
\vdots & & \\
2n-1 & \models & p, \text{ for } n \geq 3 \text{ and } 2n-1 \leq i, \\
2n & \not\models & p, \text{ for } n \geq 3 \text{ and } 2n < i.
\end{array}
$$

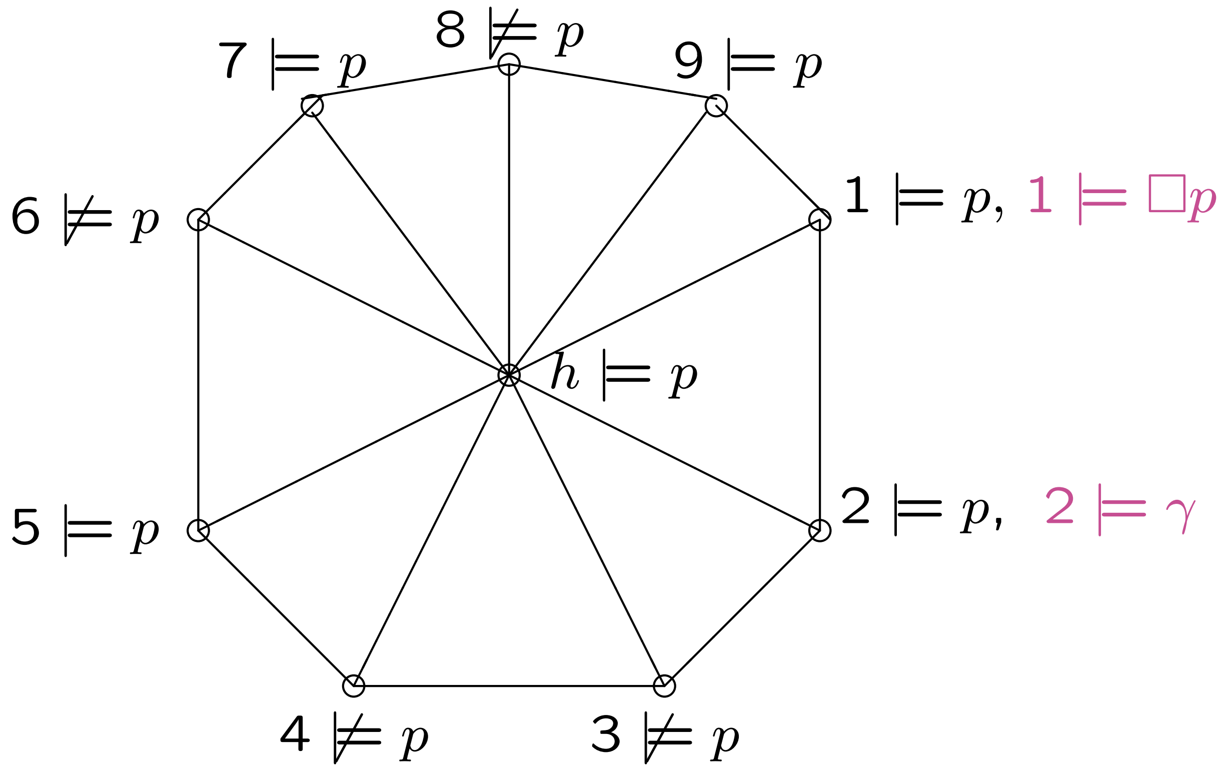
Let $k = 7$ and $i = 9$.



$8 \not\models p$
$7 \models p$
$9 \models p$
$6 \not\models p$
$1 \models p,$
$h \models p$
$5 \models p$
$2 \models p$
$4 \not\models p$
$3 \not\models p$

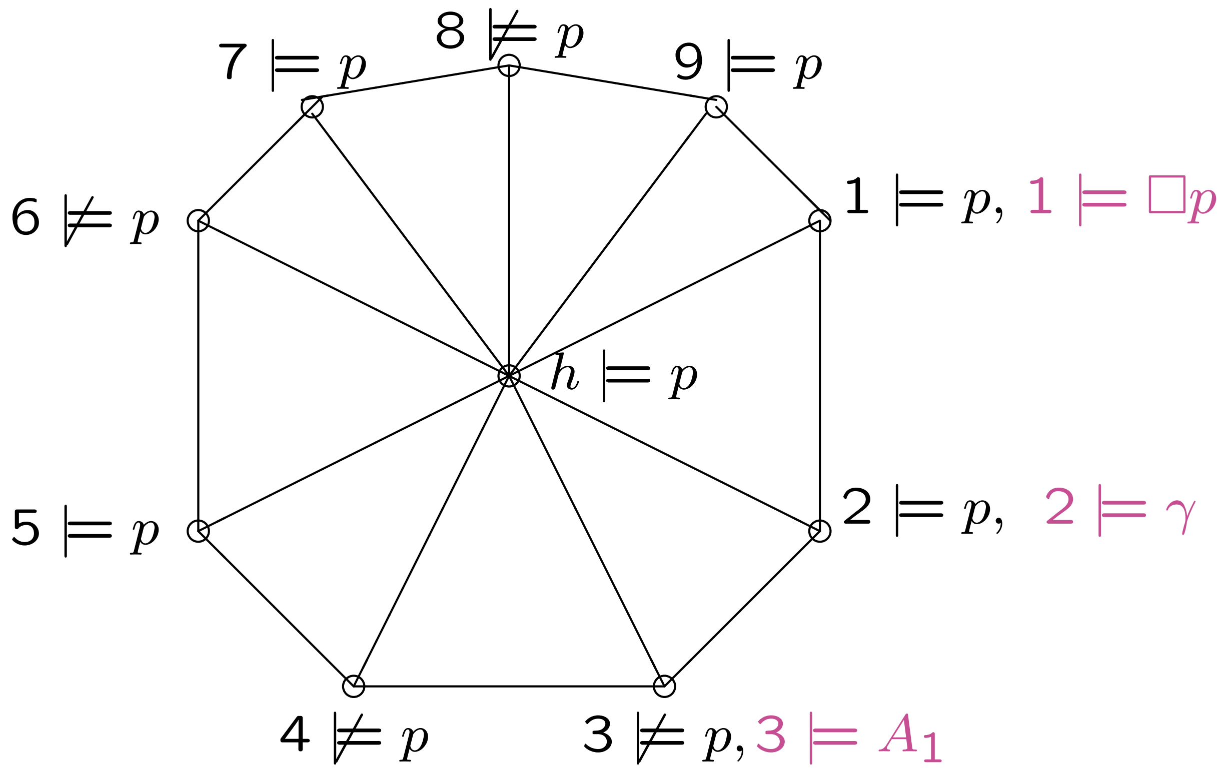


$8 \not\models p$
$7 \models p$
$9 \models p$
$6 \not\models p$
$1 \models p$, $1 \models \Box p$
$h \models p$
$5 \models p$
$2 \models p$
$4 \not\models p$
$3 \not\models p$

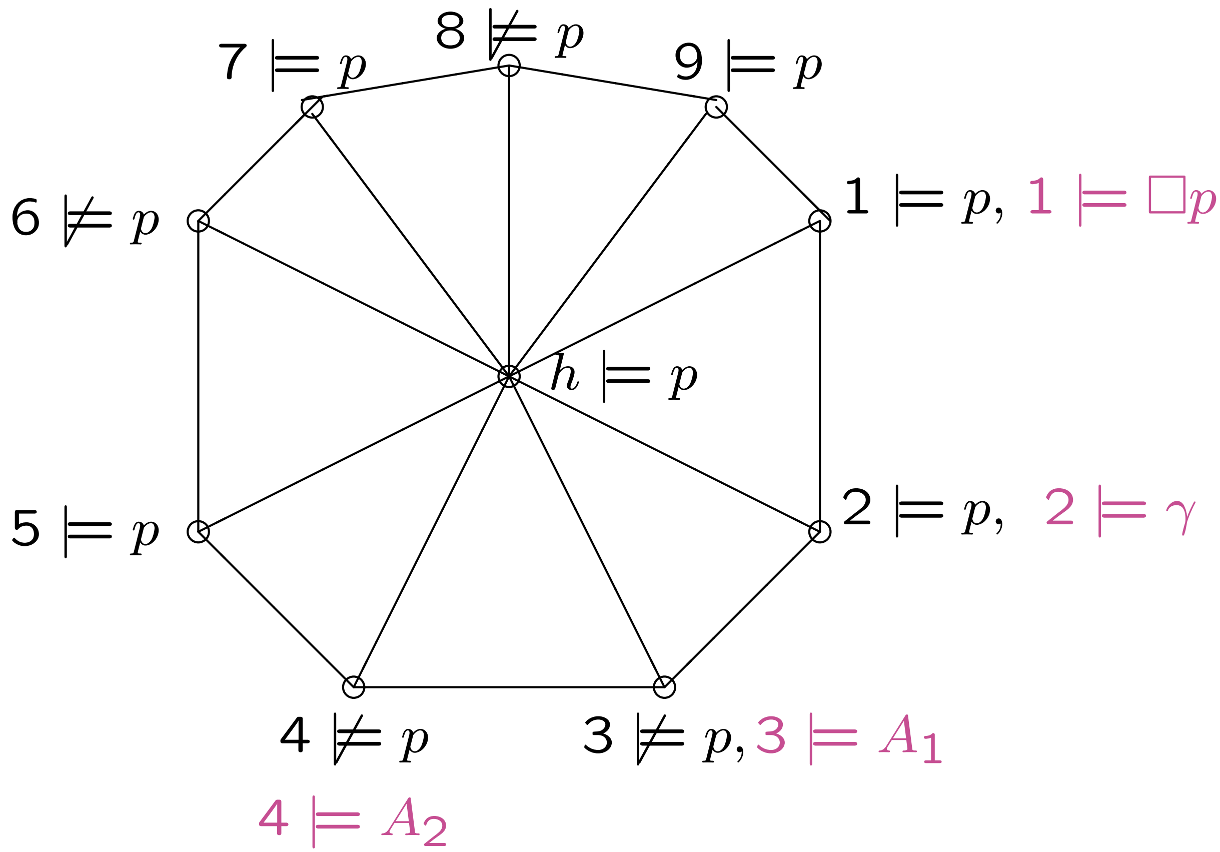


where $\gamma = \beta \wedge \Diamond A_1 \wedge \neg \Diamond A_2$

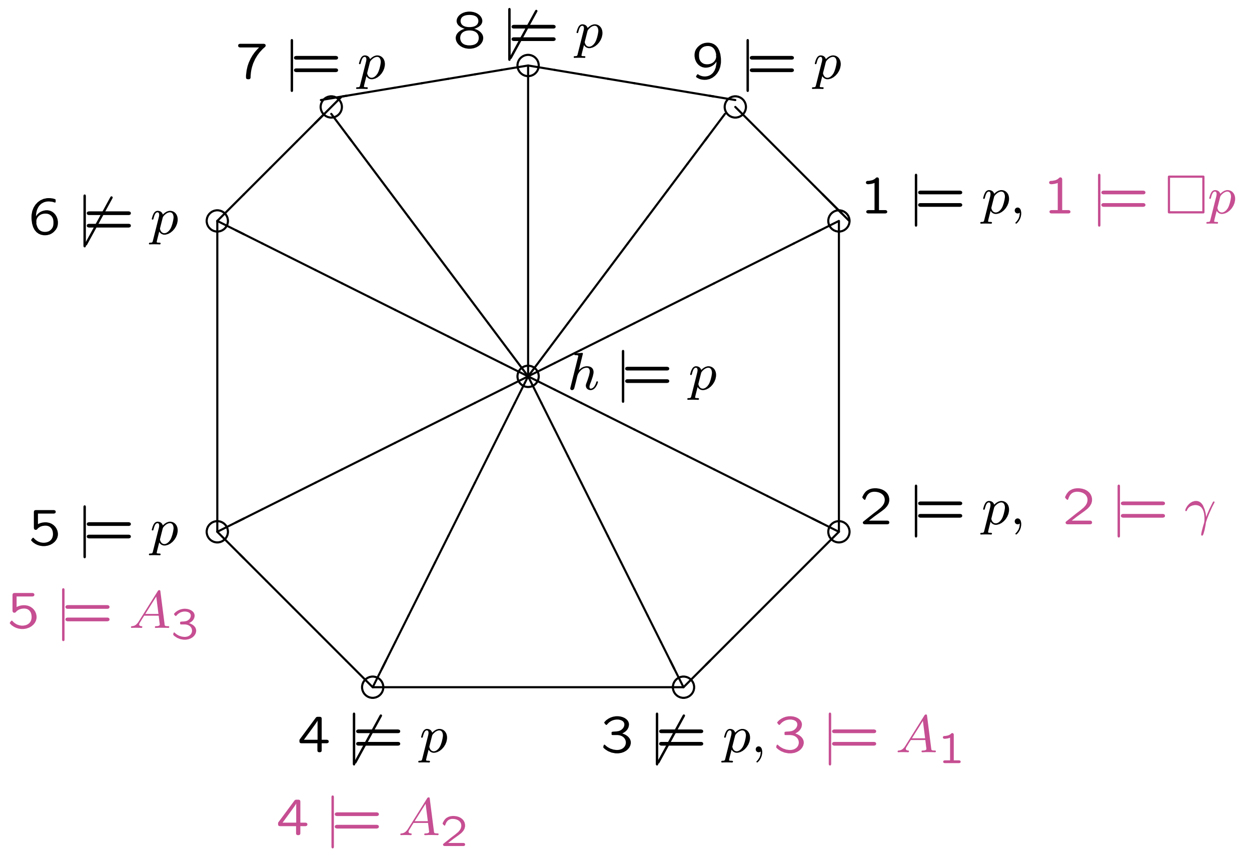
$\beta = \neg \Box p \wedge \Diamond \Box p$



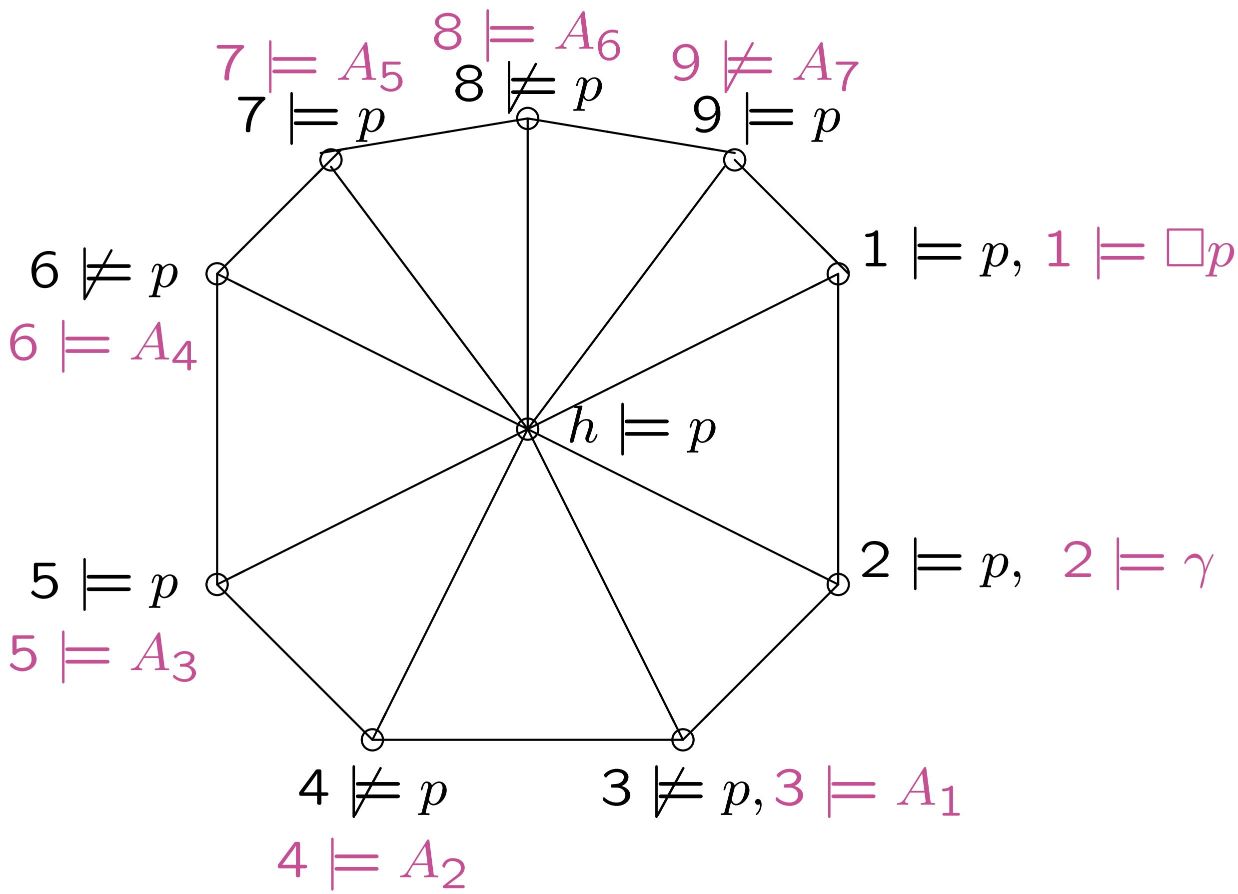
$8 \not\models p$
$7 \models p$
$9 \models p$
$6 \not\models p$
$1 \models p$, $1 \models \Box p$
$h \models p$
$5 \models p$
$2 \models p$, $2 \models \gamma$
$4 \not\models p$
$3 \not\models p$, $3 \models A_1$



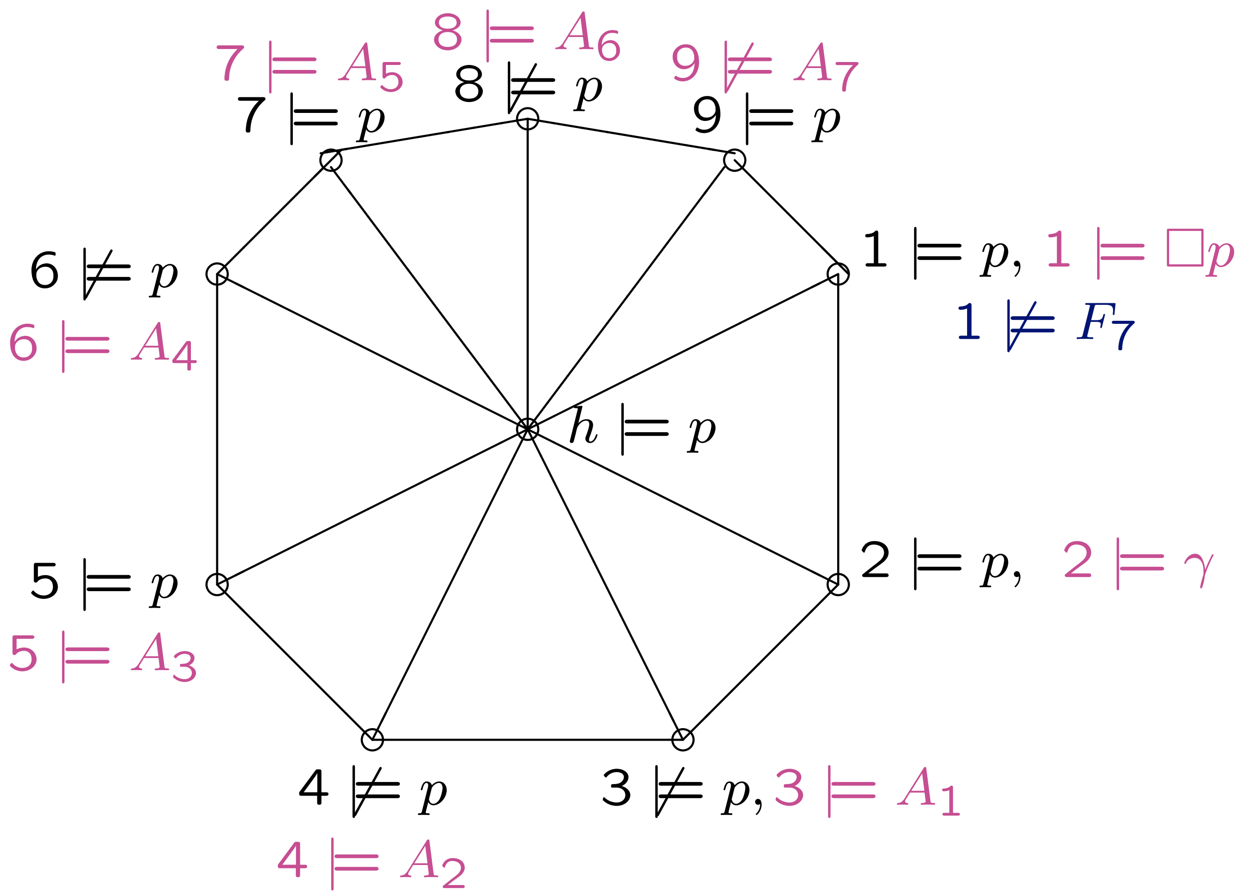
$8 \not\models p$
$7 \models p$
$9 \models p$
$6 \not\models p$
$1 \models p$, $1 \models \Box p$
$h \models p$
$5 \models p$
$2 \models p$, $2 \models \gamma$
$4 \not\models p$
$3 \not\models p$, $3 \models A_1$
$4 \models A_2$



$8 \not\models p$
$7 \models p$
$9 \models p$
$6 \not\models p$
$1 \models p$, $1 \models \Box p$
$h \models p$
$5 \models p$
$2 \models p$, $2 \models \gamma$
$5 \models A_3$
$4 \not\models p$
$3 \not\models p$, $3 \models A_1$
$4 \models A_2$



8 ⊨ A6
7 ⊨ A5
8 ⊭ p
9 ⊭ A7
7 ⊨ p
9 ⊨ p
6 ⊭ p
1 ⊨ p, 1 ⊨ □p
6 ⊨ A4
h ⊨ p
5 ⊨ p
2 ⊨ p, 2 ⊨ γ
5 ⊨ A3
4 ⊭ p
3 ⊭ p, 3 ⊨ A1
4 ⊨ A2



where $F_7 = (\Box p \wedge \bigwedge_{i=2}^{6} C_i \wedge D_6 \wedge E) \to \Diamond^2 A_7$.

Then the point 1 is the only point such that $1 \models \Box p$. And further:

$$\begin{aligned} h &\models p, \\ 2 &\models \gamma, \\ 3 &\models A_1, \\ 4 &\models A_2, \\ &\vdots \\ k+1 &\models A_{k-1}, \\ k+2 &\not\models A_k, \text{ and } k+2 \models \varepsilon \end{aligned}$$

Then we see that for all $j = 3, ..., k+1$ we have: $j \models A_n$ iff $n = j-2$. We conclude that for all $j = 3, ..., k+1$ it holds that: $j \models \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E$. Then the predecessor of the formula F_k: $(\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E)$ is true only at the point 1. At the point 1 we also have: $1 \not\models \Diamond^2 A_k$, because there is no point in the frame satisfying A_k. Hence at the point 1, the formula F_k is not true.

In the case when $i = m(k+2)$ for some $m \neq 1$, $m \in \omega$ we define the valuation similarly:

$$
\begin{array}{rcl}
h & \models & p, \\
1 + l(k+2) & \models & p, \\
2 + l(k+2) & \models & p, \\
3 + l(k+2) & \not\models & p, \\
4 + l(k+2) & \not\models & p, \\
& \vdots & \\
2n - 1 + l(k+2) & \models & p, \quad \text{for } \; n \geq 3 \; \text{ and } \; 2n-1 \leq i, \\
2n + l(k+2) & \not\models & p, \quad \text{for } \; n \geq 3 \; \text{ and } \; 2n < i.
\end{array}
$$

for all l such that: $0 \leq l \leq m$. The rest of the proof in this case proceeds analogously to the case $i = k + 2$.

($\Rightarrow$) Suppose there is a point $x \in W$ such that:

$$x \models (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E)$$
$$x \models \neg\Diamond^2 A_k.$$

First, let us observe that $x \neq h$ because $x \models \Diamond\gamma$. Let $x = 1$. Then we know that there is a point 2 such that $2 \models \gamma$ what involves existence of the next point 3 such that $3 \models A_1$. Because of C_i, $i = 1, 2, ..., k - 1$ we know that there is a sequence of points $3, 4, ..., k + 1$ such that $n \models A_{n-2}$ for $2 \leq n \leq k + 1$ and $k + 1 \models \neg\Diamond A_k$. Then the point $k + 2$ next to the point $k + 1$, has to validate the formula ε. Because $h \not\models \varepsilon$ and $k, k+1 \not\models \varepsilon$ then it must be a rim element. It has to see some point validating $\Box p$ and if it sees the point 1 then we have that $i = k+2$. But suppose that $k+2$ does not see the point 1. Anyway, it has to see another point validating $\Box p$. Say, it is the point $k+3$. But it has to be $k+3 \models \Diamond\gamma$. Because $h \not\models \gamma$ then it has to be other point, say $k+4$ such that $k+4 \models \gamma$. Then there has to be a next point $k+5$ different from h such that $k+5 \models A_1$. Again from C_i for $i = 1, 2, ..., k-1$ we have to have: $k+6 \models A_2, ..., 2k+3 \models A_{k-1}$. Then we have that there has to be a point $2k+4$ validating ε, and then some point validating $\Box p$. If it is the point 1 then we have $i = 2(k+2)$. If not, then we have analogously another sequence of $k+2$ points and so on.

$\Box$

The main theorem is the following:

Theorem 8. *There is a continuum of normal modal logics over $\mathbf{T_2}$ logic, defined by formulas written in one variable.*

Proof. Let $Prim := \{n \in \omega : n + 2 \text{ is prime, } n \geq 5\}$. Let $X, Y \subset Prim$ and $X \neq Y$. Consider logics: $L_X := \mathbf{T_2} \oplus \{F_k : k \in X\}$ and $L_Y := \mathbf{T_2} \oplus \{F_k : k \in Y\}$. From Lemma 7 we know that if $j \notin X$ then $F_j \in L_Y$. That means that we are able to define a continuum of different logics above $\mathbf{T_2}$ by formulas of one variable.

[3] Kostrzycka Z., *On the existence of a continuum of logics in NEXT(*$\mathbf{KTB} \oplus \Box^2 p \rightarrow \Box^3 p$), accepted to Bulletin of the Section of Logic.