On the existence of a continuum of logics in $\mathrm{NEXT}(\mathbf{KTB} \oplus \Box^2 p \to \Box^3 p)$

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August 2, 2007

Abstract

In this paper we consider formulas in one variable in the normal logic $\mathbf{T_2} = \mathbf{KTB} \oplus \Box^2 p \to \Box^3 p$. Next, we use the formulas to define a continuum of logics over $\mathbf{T_2}$.

1 Introduction

In this paper we investigate normal modal logics over $\mathbf{T_2} = \mathbf{KTB} \oplus \Box^2 p \to \Box^3 p$. The **KTB** logic is known as the Brouwer system and is an example of a non-transitive logic. It is characterized by the class of reflexive symmetric and non-transitive frames. There is very few results concerning this logic; some of them are included in [1] - [5].
Let us notice that adding the axiom $\Box^2 p \to \Box^3 p$ to the Brouwer logic involves the following first order condition on frames:

$$(tran_2) \quad \forall_{x,y}(\text{if } \; xR^3 y \; \text{ then } \; xR^2 y).$$

The above property is known as a two-step transitivity.

2 Formulas in one variable in $NEXT(\mathbf{T_2})$

In this section we remind ourselves the infinite sequence of non-equivalent formulas in one variable defined in [2]. Denote $\alpha := p \wedge \neg\Diamond\Box p$.

Definition 1.

$$A_1 := \neg p \wedge \Box\neg\alpha,$$
$$A_2 := \neg p \wedge \neg A_1 \wedge \Diamond A_1,$$
$$A_3 := \alpha \wedge \Diamond A_2,$$

For $n \geq 2$:

$$\begin{aligned} A_{2n} &:= \neg p \wedge \Diamond A_{2n-1} \wedge \neg A_{2n-2}, \\ A_{2n+1} &:= \alpha \wedge \Diamond A_{2n} \wedge \neg A_{2n-1}. \end{aligned}$$

Let us define the following model (see Figure 1):

Definition 2. $\mathfrak{M} = \langle W, R, V \rangle$*, where*

$$W := \{x_1, x_2\} \cup \{y_i, i \geq 1\}.$$

The relation R is reflexive, symmetric, 2-step accessible and:

$$x_1 R x_2, \quad x_2 R y_i \text{ for any } i \geq 1,$$
$$y_i R y_j \text{ if } |i-j| \leq 1 \text{ for any } i, j \geq 1.$$

The valuation is the following:

$$V(p) := \{x_1, x_2\} \cup \{y_{2m+1}, m \geq 1\}.$$

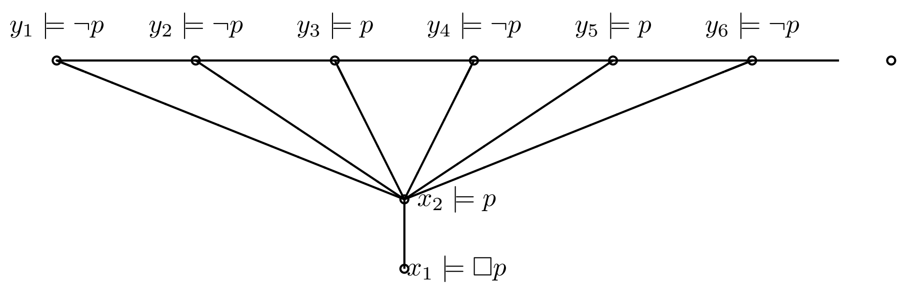

Figure 1.

Lemma 3. *For any $i \geq 1$ and for any $x \in W$ the following holds:*

$$x \models A_i \quad \textit{iff} \quad x = y_i.$$

Proof. A detailed proof is presented in [2]. □

Theorem 4. *The formulas $\{A_i\}$, $i \geq 1$ are non-equivalent in the logic* $\mathbf{T_2}$.

Proof. Obvious.

□

3 The existence of a continuum of logics over $\mathbf{T_2}$

Yutaka Miyazaki in [4] proved the existence of a continuum of logics over **KTB**. First, he showed the existence of a continuum of orthologics and then applied an embedding from orthologics to **KTB** logics. In [5] Y. Miyazaki proved the existence of a continuum of logics over $\mathbf{T_2}$. In the proof he considered logics determined by the so-called wheel frames.

Definition 5. *For $n \in \omega$, $n \geq 5$, the wheel frame $\mathfrak{W}_n = \langle W, R \rangle$ of degree n consists of the following set and binary relation: $W = rim(W) \cup h$, where $rim(W) := \{1, 2, ..., n\}$ and $h \notin rim(W)$. Any element in $rim(W)$ is called a rim element, whereas the element h - the hub element. The relation R is defined as: $R := \{(x, y) \in (rim(W))^2 : |x - y| \leq 1(mod(n-1))\} \cup \{(h, h)\} \cup \{(h, x), (x, h) : x \in rim(W)\}$.*

For example in Figure 2 we present a diagram of the $\mathfrak{W}_8$.

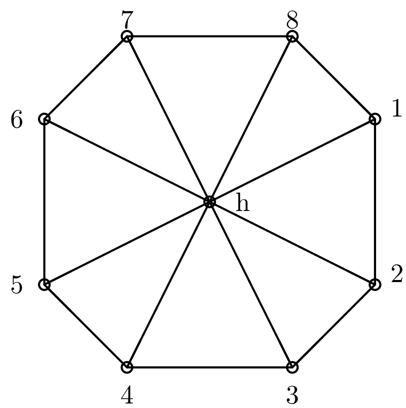

Figure 2.

Y.Miyazaki proved the following two lemmas (Proposition 19, Lemma 20 from [5]):

Lemma 6. *For $m > n \geq 5$, $L(\mathfrak{W}_n) \not\subseteq L(\mathfrak{W}_m)$.*

Lemma 7. *For $m \geq n \geq 5$, suppose there is a p-morphism from $\mathfrak{W}_m$ to $\mathfrak{W}_n$. Then m is divisible by n.*

On the base of these two lemmas and by using the splitting technique effectively, Y. Miyazaki constructed a continuum of normal modal logics over $\mathbf{KTB} \oplus \Box^2 p \to \Box^3 p$ logic.
Below, we present an improved method (in comparison to the one from [2]) for obtaining a continuum of logics above $\mathbf{T_2}$. Actually, we axiomatize the logics determined by wheel frames with formulas in one variable. Let us define new formulas for $k \in \omega$:

Definition 8.

$$
\begin{aligned}
C_k &:= \Box^2[A_{k-1} \to \Diamond A_k], \quad \textit{for} \quad k > 2, \\
D_k &:= \Box^2[(A_k \wedge \neg\Diamond A_{k+1}) \to \Diamond\varepsilon], \\
E &:= \Box^2(\Box p \to \Diamond\gamma), \\
F_k &:= (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E) \to \Diamond^2 A_k, \\
\textit{where} & \\
\beta &:= \neg\Box p \wedge \Diamond\Box p, \\
\gamma &:= \beta \wedge \Diamond A_1 \wedge \neg\Diamond A_2, \\
\varepsilon &:= \beta \wedge \neg\Diamond A_1 \wedge \neg\Diamond A_2.
\end{aligned}
$$

Lemma 9. *Let $k \geq 5$ and k be an odd number. Then $\mathfrak{W}_i \not\models F_k$ iff i is divisible by $k+2$.*

Proof. ($\Leftarrow$)
Let $i = k+2$. We define the following valuation in the frame $\mathfrak{W}_i$:

$$\begin{array}{rcl} h & \models & p, \\ 1 & \models & p, \\ 2 & \models & p, \\ 3 & \not\models & p, \\ 4 & \not\models & p, \\ 2n-1 & \models & p, \text{ for } n \geq 3 \text{ and } 2n-1 \leq i, \\ 2n & \not\models & p, \text{ for } n \geq 3 \text{ and } 2n < i. \end{array}$$

Then the point 1 is the only point such that $1 \models \Box p$. And further:

$$\begin{array}{rcl} h & \models & p, \\ 2 & \models & \gamma, \\ 3 & \models & A_1, \\ 4 & \models & A_2, \\ k+1 & \models & A_{k-1}, \\ k+2 & \not\models & A_k, \text{ and } k+2 \models \varepsilon \end{array}$$

Then we see that for all $j = 3, ..., k+1$ we have: $j \models A_n$ iff $n = j-2$. We conclude that for all $j = 1, 2, ..., k+1$ it holds that: $j \models \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E$. Then the predecessor of the formula F_k: $(\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E)$ is true only at the point 1. At the point 1 we also have: $1 \not\models \Diamond^2 A_k$, because there is no point in the frame satisfying A_k. Hence at the point 1, the formula F_k is not true.
In the case when $i = m(k+2)$ for some $m \neq 1$, $m \in \omega$ we define the valuation similarly:

$$\begin{array}{rcl} h & \models & p, \\ 1+l(k+2) & \models & p, \\ 2+l(k+2) & \models & p, \\ 3+l(k+2) & \not\models & p, \\ 4+l(k+2) & \not\models & p, \\ 2n-1+l(k+2) & \models & p, \text{ for } n \geq 3 \text{ and } 2n-1+l(k+2) \leq i, \\ 2n+l(k+2) & \not\models & p, \text{ for } n \geq 3 \text{ and } 2n+l(k+2) < i. \end{array}$$

for all l such that: $0 \leq l < m$. The rest of the proof in this case proceeds analogously to the case $i = k+2$.
($\Rightarrow$) Suppose there is a point $x \in W$ such that:

$$\begin{array}{rcl} x & \models & (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E) \\ x & \models & \neg \Diamond^2 A_k. \end{array}$$

First, let us observe that $x \neq h$ because if $h \models \Box p$ then for all $x \in rim(W)$ we have $x \models p$. Hence there is no point $x' \in rim(W)$ such that $x' \models \gamma$. Then it is impossible that at the point h formula $(\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E)$ is true. Then x has to belong to $rim(W)$. Let $x = 1$. Then we know that there is a point 2 such that $2 \models \gamma$ what involves existence of the next point 3 such that $3 \models A_1$. Because of C_i, $i = 1, 2, ..., k-1$ we know that there is a sequence of points $3, 4, ..., k+1$

such that $n \models A_{n-2}$ for $2 \leq n \leq k+1$ and $k+1 \models \neg\Diamond A_k$. Then a point $k+2$, which is next to the point $k+1$, has to validate the formula ε. Because $h \not\models \varepsilon$ and $k, k+1 \not\models \varepsilon$ then it must be a new rim element. It has to see some point validating $\Box p$ and if it sees the point 1 then we have that $i = k+2$. But suppose that $k+2$ does not see the point 1. Anyway, it has to see another point validating $\Box p$. Say, it is the point $k+3$. But it has to be $k+3 \models \Diamond\gamma$. Because $h \not\models \gamma$ then it has to be other point, say $k+4$ such that $k+4 \models \gamma$. Then there has to be a next point $k+5$ different from h such that $k+5 \models A_1$. Again from C_i for $i = 1, 2, ..., k-1$ we have to have: $k+6 \models A_2, ..., 2k+3 \models A_{k-1}$. Then we have that there has to be a point $2k+4$ validating ε, and then some point validating $\Box p$. If it is the point 1 then we have $i = 2(k+2)$. If not, then we have analogously another sequence of $k+2$ points and so on.

□

The main theorem is the following:

Theorem 10. *There is a continuum of normal modal logics over* $\mathbf{T_2}$ *defined by formulas written in one variable.*

Proof. Let $Prim := \{n \in \omega : n+2 \text{ is prime}, \ n \geq 5\}$. Let $X, Y \subset Prim$ and $X \neq Y$. (Exactly: $X \not\subseteq Y$ and $Y \not\subseteq X$). Consider logics: $L_X := \mathbf{T_2} \oplus \{F_k : k \in X\}$ and $L_Y := \mathbf{T_2} \oplus \{F_k : k \in Y\}$. Let $j \in Y \setminus X$. Obviously: $F_j \in L_Y$. From Lemma 9 we know that $F_j \notin L_X$, because $W_{j+2} \not\models F_j$ and $\forall_{i \in X} \ [i \neq j \quad \Rightarrow \quad W_{j+2} \models F_i]$. That means that we are able to define a continuum of different logics above $\mathbf{T_2}$ by formulas of one variable.

□

References

[1] Byrd M., *On the addition of weakened L-reductionaxioms to the Brouwer system*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 24, 1978, 405-408.

[2] Kostrzycka Z., *On formulas in one variable in $NEXT(KTB)$*, Bulletin of the Section of Logic, Vol.35:2/3, (2006), 119-131.

[3] Makinson D., *Non-equivalent formulae in one variable in a strong omnitemporal modal logic*, Zeitchr. f. math. Logik und Grundlagen d. Math., Vol.27, (1981), 111-112.

[4] Miyazaki Y. *Binary Logics, Orthologics and their Ralations to Normal Modal Logics*, Advances in Modal Logic, Vol.4, (2003), 313-333.

[5] Miyazaki Y. *Normal modal logics containing KTB with some finiteness conditions*, Advances in Modal Logic, Vol.5, (2005), 171-190.