CORRECTION OF THE ARTICLE: ON LINEAR BROUWERIAN LOGICS

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1. A shortcut

Definition 1. *We say that a frame $\langle W, R\rangle$ consists of linearly ordered blocks if the following two conditions hold:*

$(L1)$ $B_1 \cap B_2 \cap B_3 = \emptyset$,

$(L2)$ $(B_1 \cap B_2 \neq \emptyset \ \& \ B_2 \cap B_3 \neq \emptyset) \quad \Rightarrow \quad (B_1 \cap B_2) \cup (B_2 \cap B_3) = B_2$

for any three distinct blocks B_1, B_2, B_3

It occurred that the generalized notion of linearity for reflexive and symmetric structures has an adequate syntactic characterization [1]. The following formula is given there:

$$(3') \ := \Box p \vee \Box(\Box p \to \Box q) \vee \Box((\Box p \wedge \Box q) \to r).$$

and the logic: $\mathbf{KTB.3'} := \mathbf{KTB} \oplus (3')$ is considered. In [1], it is also proven that:

Theorem 1. *[Theorem 3.1 from [1]] Logic $\mathbf{KTB.3'}$ is complete with respect to the class of reflexive and symmetric frames with linearly ordered blocks. Logic $\mathbf{KTB.3'}$ has f.m.p.*

THERE IS A MISTAKE IN THE ABOVE THEOREM. BELOW I GIVE ITS CORRECTION.

WE ADD A NEW AXIOM:

$$(A) \ := \Box((\Box p \wedge q) \to r) \vee \Box((\Box q \wedge r) \to s) \vee \Box((\Box r \wedge s \wedge \Diamond\neg s) \to p) \vee$$
$$\vee\Box((\Box s \wedge p \wedge \Diamond\neg p) \to q).$$

AND DEFINE THE LOGIC: $\mathbf{KTB.3'A} := \mathbf{KTB} \oplus (3') \oplus (A)$.

Theorem 2. *Logic $\mathbf{KTB.3'A}$ is complete with respect to the class of reflexive and symmetric frames with linearly ordered blocks.*

Proof. First we prove the soundness of $\mathbf{KTB.3'A}$ with respect to the appropriate relational structures:

If $\ \vdash_{KTB.3'A} \alpha\,$, then $\ \mathfrak{F} \models \alpha$; for any Kripke frame $\mathfrak{F} \in \mathcal{LOB}$ and any formula α.

We need to show that in each reflexive, symmetric and linearly ordered Kripke frame, the formulas (3′) and (A) are valid. Suppose, on the contrary, that there is a frame $\mathfrak{F} = \langle W, R\rangle$ from $\mathcal{LOB}$ such that $\mathfrak{F} \not\models (3')$. Then there is a point $x \in W$ such that: $x \not\models \Box p \vee \Box(\Box p \to \Box q) \vee \Box((\Box p \wedge \Box q) \to r)$. Hence

(1) $x \not\models \Box p$ and
(2) $x \not\models \Box(\Box p \to \Box q)$, and
(3) $x \not\models \Box((\Box p \wedge \Box q) \to r)$.

From (1) we conclude that there is a point $y_1 \in W$, xRy_1 such that $y_1 \not\models p$.

From (2) we conclude that there is a point $y_2 \in W$, xRy_2 such that $y_2 \not\models \Box p \to \Box q$. Then $y_2 \models \Box p$ and $y_2 \not\models \Box q$. Because $y_1 \not\models p$ and $y_2 \models \Box p$ then we conclude that $\neg y_2Ry_1$. Because xRy_2 and R is symmetric we get $x \models p$ (hence $y_1 \neq x$ and $y_2 \neq x$). Since $y_2 \not\models \Box q$ then there is also a point $y_3 \in W$, y_2Ry_3 such that $y_3 \not\models q$ (y_3 does not have to be distinct from y_2).

From (3) we get that there is a point $y_4 \in W$, xRy_4 such that $y_4 \not\models (\Box p \wedge \Box q) \to r$. Hence $y_4 \models \Box p \wedge \Box q$ and $y_4 \not\models r$. Then $y_4 \models \Box p$ and $y_4 \models \Box q$. We will show that $y_4 \neq y_i$ for $i = 1, 2, 3$ and $y_4 \neq x$.

- Because $y_4 \models \Box p$ then from reflexivity of R we get $y_4 \models p$. We know that $y_1 \not\models p$. Then $y_4 \neq y_1$.
- Since $y_4 \models \Box q$ and $y_2 \not\models \Box q$ then $y_4 \neq y_2$.
- Suppose that $y_3 \neq y_2$. We know that $y_4 \models \Box q$ and $y_3 \not\models q$. By reflexivity of R we conclude that $y_4 \neq y_3$.
- We have assume that $x \not\models \Box p$ and $y_4 \models \Box p$. Then $y_4 \neq x$.

Further we will show that $\neg y_4Ry_i$ for $i = 1, 3$ (under the assumption that $y_3 \neq y_2$) and $x \models q$ and hence $y_3 \neq x$.

- Suppose, on the contrary, that y_4Ry_1. Since $y_4 \models \Box p$ then $y_1 \models p$. But this is a contradiction.
- Suppose, on the contrary, that y_4Ry_3. Because $y_4 \models \Box q$ then we would obtain $y_3 \models q$. A contradiction.
- Because $y_4 \models \Box q$ and xRy_4 (and by symmetry of R) we get $x \models q$. Then $x \neq y_3$.

The following two situations are possible:

- y_4Ry_2. Then $y_2 \models q$, so $y_3 \neq y_2$. We get a frame with a sub-frame identical with the one presented in Diagram 1. The condition (L2) of linearity fails in this case.
- $\neg y_4Ry_2$. Then $\{x, y_4\} \subset B_1$, $\{x, y_1\} \subset B_2$ and $\{x, y_2\} \subset B_3$ and $y_4 \notin B_3$, $y_2 \notin B_1$, $y_1 \notin B_1$, for three blocks B_1, B_2 and B_3. Hence the blocks are distinct. The situation is similar to

the one presented in Diagram 2. We get $x \in B_1 \cap B_2 \cap B_3$. The condition (L1) of linearity fails. It is not important here, if $y_3 R x$ or $y_3 R y_1$.

Let us add that if $y_3 = y_2$ then $y_2 \not\models q$. Then $\neg y_4 R y_2$ (since $y_4 \models \Box q$). As above, the condition (L1) of linearity fails.

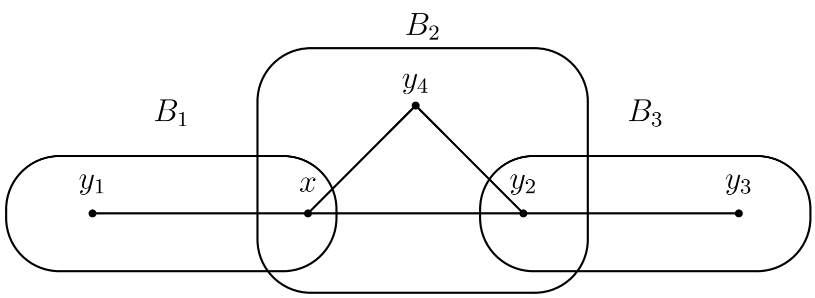


Diagram 1.

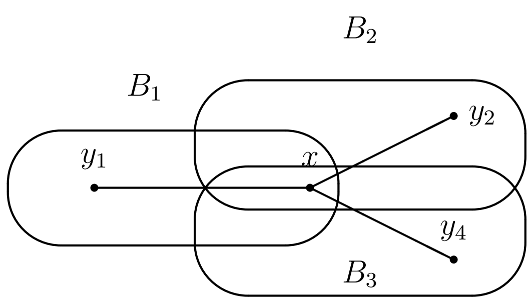


Diagram 2.

Suppose then that the formula (A) fails in some $\mathfrak{F} = \langle W, R \rangle \in \mathcal{LOB}$. There is a point $x \in W$ such that

(1) $x \not\models \Box((\Box p \wedge q) \to r)$ and
(2) $x \not\models \Box((\Box q \wedge r \wedge) \to s)$, and
(3) $x \not\models \Box((\Box r \wedge s \wedge \Diamond\neg s) \to p)$, and
(4) $x \not\models \Box((\Box s \wedge p \wedge \Diamond\neg p) \to q)$

From (1) we conclude that there is a point $y_1 \in W$, xRy_1 such that $y_1 \not\models (\Box p \wedge q) \to r)$. Hence $y_1 \models \Box p$, $y_1 \models q$ and $y_1 \not\models r$.

Similarly from (2) we conclude that there is a point $y_2 \in W$, xRy_2 such that $y_2 \not\models (\Box q \wedge r) \to s)$. Hence $y_2 \models \Box q$, $y_2 \models r$, and $y_2 \not\models s$.

Obviously, $y_2 \neq y_1$.

Similarly from (3) we conclude that there is a point $y_3 \in W$, xRy_3 such that $y_3 \not\models (\Box r \wedge s \wedge \Diamond\neg s) \to p)$. Hence $y_3 \models \Box r$, $y_3 \models s$, $y_3 \models \Diamond\neg s$ and $y_3 \not\models p$.

Obviously, $y_3 \neq y_1$ and $y_3 \neq y_2$.

Similarly from (4) we conclude that there is a point $y_4 \in W$, xRy_4 such that $y_4 \not\models (\Box s \wedge p \wedge \Diamond\neg p) \to q)$. Hence $y_4 \models \Box s$, $y_4 \models p$ and $y_4 \models \Diamond\neg p$ and $y_4 \not\models q$.

Obviously: $y_4 \neq y_i$ for $i = 1, 2, 3$.

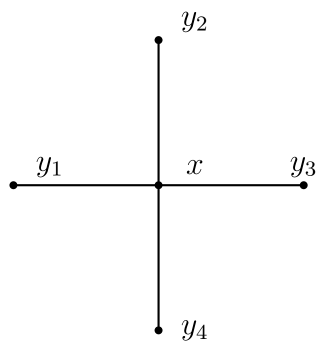


Diagram 3

Further we notice that: $\neg y_1 R y_3$ and $\neg y_2 R y_4$ (and symmetrically $\neg y_3 R y_1$ and $\neg y_4 R y_2$) because $y_1 \models \Box p$ and $y_3 \not\models p$ as well as $y_2 \models \Box q$ and $y_4 \not\models q$. Since

$$(*)\ y_2 \models \Diamond\neg s$$
$$(**)\ y_4 \models \Diamond\neg p$$

then the following four situations are possible:

(1) $y_2 R y_3$ and $y_3 R y_4$ (then (*) and (**) are fulfilled). Because $\neg y_1 R y_3$ and $\neg y_2 R y_4$ then exist three distinct blocks B_1, B_2, B_3 such that $\{x, y_2, y_3\} \subset B_1$, $\{x, y_3, y_4\} \subset B_2$ and $\{x, y_1\} \subset B_3$. Also $y_2 \notin B_2$, $y_4 \notin B_1$ and $y_1 \notin B_1 \cup B_2$. Then $B_1 \cap B_2 \cap B_3 \neq \emptyset$. A contradiction with (L1). See Diagram 4.

(2) $y_2 R y_3$ and $\neg y_3 R y_4$ (then (*) is fulfilled). To fulfil (**) we need a new point z such that zRy_4 and $z \not\models p$. Obviously $\neg z R y_1$.

   (a) Suppose that $\neg y_1 R y_4$. Then there exist three distinct blocks B_1, B_2, B_3 such that $\{x, y_2, y_3\} \subset B_1$, $\{x, y_4\} \subset B_2$ and $\{x, y_1\} \subset B_3$. Also $y_2 \notin B_2$, $y_4 \notin B_1$ and $y_1 \notin B_1 \cup B_2$. Then $x \in B_1 \cap B_2 \cap B_3$. A contradiction with (L1). It does not depend on wheatear zRx or $\neg zRx$ nor wheatear zRy_i or $\neg zRy_i$ for $i = 2, 3$. See Diagram 5.

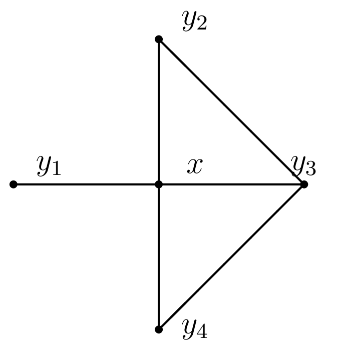


Diagram 4

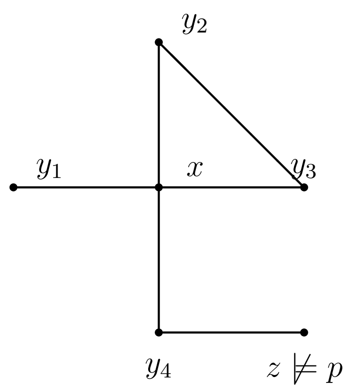


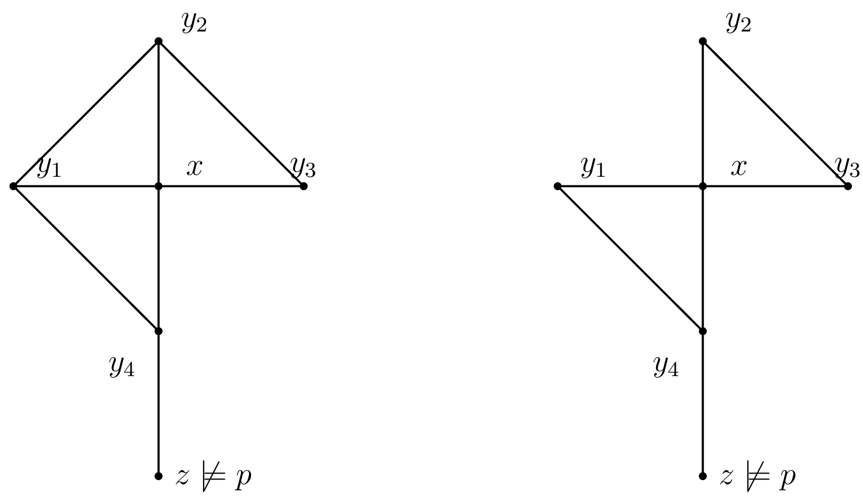
Diagram 5

(b) Suppose that y_1Ry_4.
   (i) If y_1Ry_2 then there exist three distinct blocks B_1, B_2, B_3 such that $\{x, y_1, y_2\} \subset B_1$, $\{x, y_2, y_3\} \subset B_2$ and $\{x, y_1, y_4\} \subset B_3$.Then $B_1 \cap B_2 \cap B_3 \neq \emptyset$. A contradiction with (L1). See Diagram 6a.
   (ii) If $\neg y_1Ry_2$, then there are three blocks B_1, B_2, B_3 such that $\{x, y_1, y_4\} \subset B_1$, $\{x, y_2, y_3\} \subset B_2$ and $\{z, y_4\} \subset B_3$. Then (L2) does not hold. See Diagram 6b.

(3) if $\neg y_2Ry_3$ and y_3Ry_4 then the situation is analogous to the previous one,

(4) $\neg y_2Ry_3$ and $\neg y_3Ry_4$. Then there must exist two points: zRy_4 and wRy_3 such that $z \not\models p$ and $w \not\models s$. But then there exist three blocks B_1, B_2, B_3 such that $\{x, y_2\} \subset B_1$, $\{x, y_3\} \subset B_2$ and $\{x, y_4\} \subset B_3$. They are distinct because $y_3 \notin B_1$ and $y_2 \notin B_2$ and $y_3 \notin B_3$ and $y_2 \notin B_3$. And we get $B_1 \cap B_2 \cap B_3 \neq \emptyset$. A contradiction with (L1). See Diagram 7.

The proof of completeness will be provide by Henkin's method. However, it will be provided on the base of the proof of Lemma 1 (below).



Diagrams 6a and 6b

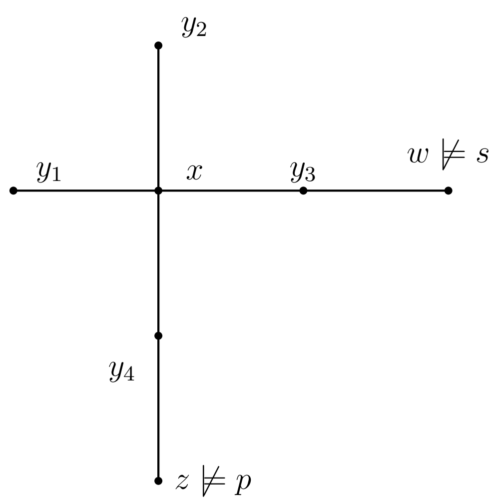


Diagram 7

$\square$

Observation 1. *B is a block iff* $\forall_x[(x \in B) \ \Leftrightarrow \ \forall_{y \in B}(xRy)]$.

Corollary 1. *If $B_1 \neq B_2$ then there exist $x_1 \in B_1$ and $x_2 \in B_2$ such that $\neg x_1 R x_2$.*

Lemma 1. *Let $\mathfrak{F}$ be a Kripke frame such that $\mathfrak{F} \models \mathbf{T}, \mathbf{B}, (3'), (A)$. Then $\mathfrak{F}$ is reflexive and symmetric and the conditions (L1) and (L2) hold.*

Proof. Obviously, if in a given Kripke frame there exists a point which is irreflexive, then the axiom **T** is falsified. Also, if in a frame exist two points being in a relation which is not symmetric, then axiom **B** is falsified. Suppose that the condition $(L1)$ does not hold in some reflexive and symmetric Kripke frame $\mathfrak{F} = \langle W, R\rangle$. Then there are three distinct block B_1, B_2 and B_3 such that $x_0 \in B_1 \cap B_2 \cap B_3$ for some x_0.

We will consider the following two exclusive possibilities:

(1) There are $x_1 \in B_1$, $x_2 \in B_2$, $x_3 \in B_3$ and $\neg x_1 R x_2$, $\neg x_1 R x_3$ and $\neg x_2 R x_3$. See Diagram 8.

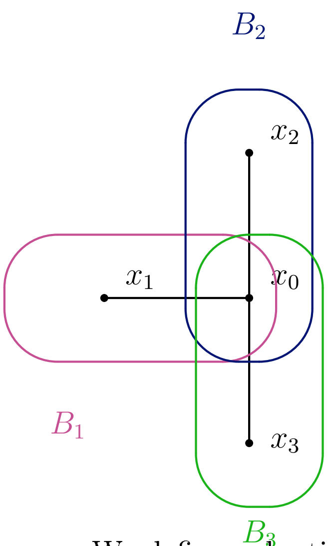


Diagram 8

We define valuation:

$$V(p) = W \setminus \{x_1\} \quad \text{and} \quad V(q) = W \setminus \{x_2\} \quad \text{and} \quad V(r) = W \setminus \{x_3\}.$$

Then we get:

$$x_2 \models_V \Box p, \;\; x_3 \models_V \Box p \wedge \Box q, \;\; \text{and} \;\; x_2 \not\models_V \Box p \to \Box q, \;\; x_3 \not\models_V (\Box p \wedge \Box q) \to r.$$

Hence $x_0 \not\models_V \Box p$, $x_0 \not\models_V \Box(\Box p \to \Box q)$ and $x_0 \not\models_V \Box[(\Box p \wedge \Box q) \to r]$. And $x_0 \not\models_V (3')$.

(2) Suppose that (1) does not hold. Without losing generality we may assume that

$$(*) \qquad B_1 \subseteq B_2 \cup B_3.$$

Since $B_1 \neq B_3$ then there exists an $x_1 \in B_1$ such that $x_1 \notin B_3$. By (*), $x_1 \in B_2$ and there exists another $x_3 \in B_3$ such that $\neg x_1 R x_3$.

Subcase (2a). Suppose that there exists $x_2 \in B_2$ such that $x_2 \neq x_1$ and $\neg x_2 R x_3$. See Diagram 9.

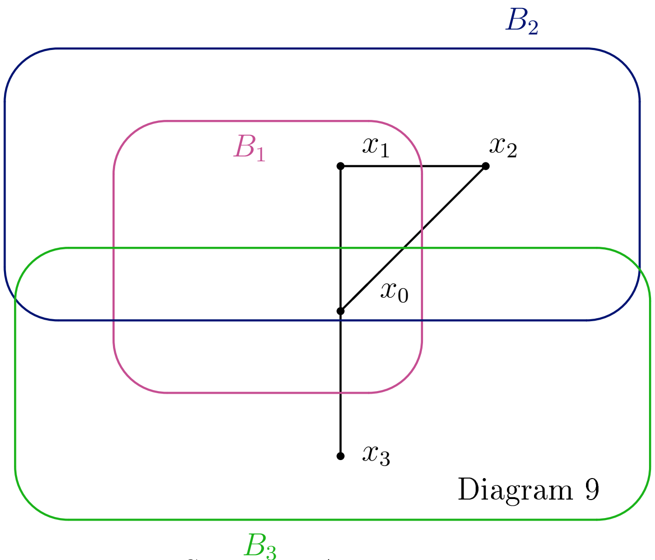


Diagram 9

Since $B_1 \neq B_2$ then there exists $y \in B_1 \setminus B_2$. By (*) we have $y \in B_3$ and $y R x_1$, see Diagram 10.

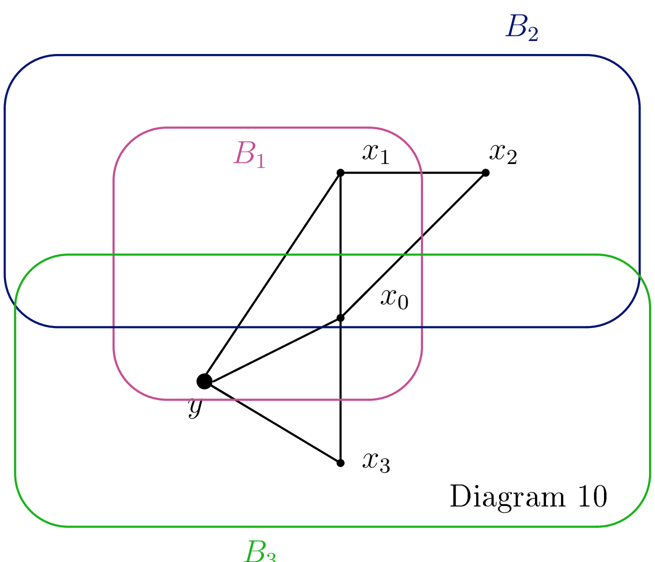


Diagram 10

We take the following valuation

$$V(p) = W \setminus \{x_2\}, \quad \text{and} \;\; V(q) = W \setminus \{x_1\} \;\; \text{and} \quad \text{and} \;\; V(r) = W \setminus \{x_3\}.$$

Then we get:

$$y \models_V \Box p, \quad x_3 \models_V \Box p \wedge \Box q, \quad \text{and} \quad y \not\models_V \Box p \to \Box q, \quad x_3 \not\models_V (\Box p \wedge \Box q) \to r.$$

Hence $x_0 \not\models_V \Box p$, $x_0 \not\models_V \Box(\Box p \to \Box q)$ and $x_0 \not\models_V \Box[(\Box p \wedge \Box q) \to r]$. And $x_0 \not\models_V (3')$.

Subcase (2b). Suppose that for any $x_2 \in B_2$ if $x_2 \neq x_1$ then $x_2 R x_3$. Obviously, $x_2 R x_1$.

Since $B_1 \neq B_2$ then there exists $y \in B_1 \setminus B_2$. By (*) we have $y \in B_3$, and $y R x_1$, and $y R x_3$. See Diagram 11.

Formula (3') is true for any valuation. But we falsify formula (A) as follows:

$$V(p) = W \setminus \{y\}, \quad \text{and} \quad V(q) = W \setminus \{x_3\}$$
$$V(r) = W \setminus \{x_2\} \quad \text{and} \quad V(s) = W \setminus \{x_1\}.$$

Then $x_2 \models_V \Box p \wedge q$, $x_1 \models_V \Box q \wedge r$, $x_3 \models_V \Box s \wedge p \wedge \Diamond\neg p$, $y \models_V \Box r \wedge s \wedge \Diamond \neg s$. Also:

$$x_1 \not\models_V (\Box q \wedge r) \to s,$$
$$x_2 \not\models_V (\Box p \wedge q) \to r,$$
$$x_3 \not\models_V (\Box s \wedge p \wedge \Diamond\neg p) \to q,$$
$$y \not\models_V (\Box r \wedge s \wedge \Diamond\neg s) \to p.$$

And $x_0 \not\models_V (A)$.

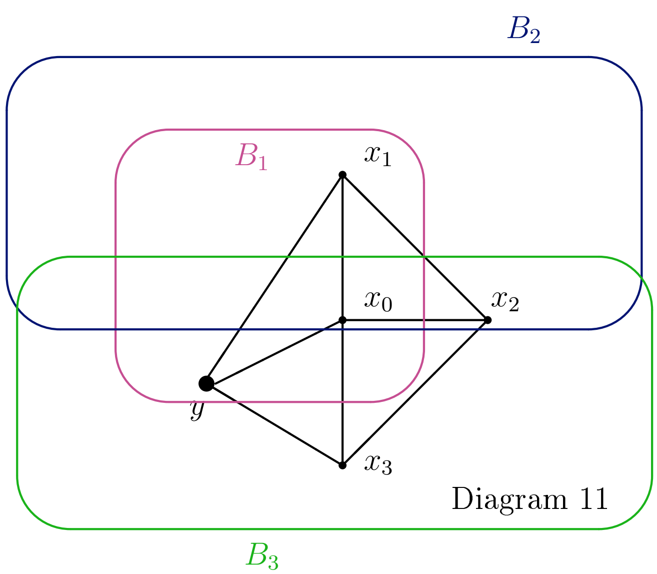


Diagram 11

Suppose, on the contrary, that the condition $(L2)$ does not hold in some Kripke frame $\mathfrak{F} = \langle W, R \rangle$. Hence there exists at least five points

x_0, x_1, x_2, x_3, x_4 belonging to three different blocks, i.e. $\{x_1, x_3\} \subset B_1$, $\{x_0, x_1, x_2\} \subset B_2$, $\{x_4, x_5\} \subset B_3$ and $x_3 \notin B_1$ as well as $x_3 \notin B_3$. Then $x_3 \notin (B_1 \cap B_2) \cup (B_2 \cap B_3)$ and $(B_1 \cap B_2) \cup (B_2 \cap B_3) \neq B_2$. See Diagram 2 from $correction_kostrzycka2.pdf$ where $x_1 := y_1$, $x_2 := x$, $x_3 := y_4$, $x_4 := y_2$, $x_5 := y_3$.

We assume that $\neg x_3 R x_1$ and $\neg x_3 R x_5$. We consider the following situations:

(1) $x_1 R x_5$.

(a) If $x_2 R x_5$ and $x_1 R x_4$, then actually points x_1, x_2, x_4, x_5 belong to the same block, hence there must exist, for example in B_1 other point $z \in B_1$ such that: zRx_1, zRx_2 and $\neg zRx_5$. Now, we take $\{x_1, x_2, z\} \subset B_1$ and $\{x_1, x_2, x_4, x_5\} \subset B_3$. We define a valuation:

$$V(p) = W \setminus \{x_3\} \text{ and } V(q) = W \setminus \{z\}, \text{ and } V(r) = W \setminus \{x_5\}$$

Then $x_5 \models_V \Box p \wedge \Box q$, $x_1 \models_V \Box p$. Also:

$$x_1 \not\models_V \Box p \to \Box q,$$
$$x_5 \not\models_V (\Box p \wedge \Box q) \to r.$$

And $x_2 \not\models_V (3')$. This valuation is independent of the assumption if zRx_4 or $\neg zRx_4$. See Diagram 12.

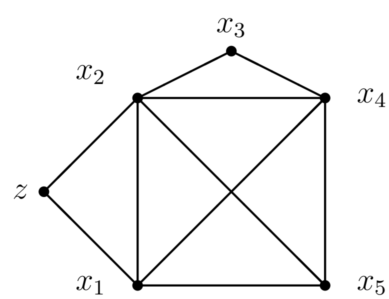


Diagram 12

(b) If $x_2 R x_5$ and $\neg x_1 R x_4$. See Diagram 13. The case is analogous to 2a as in Diagram 10.

(c) If $\neg x_2 R x_5$ and $x_1 R x_4$, then see the above case.

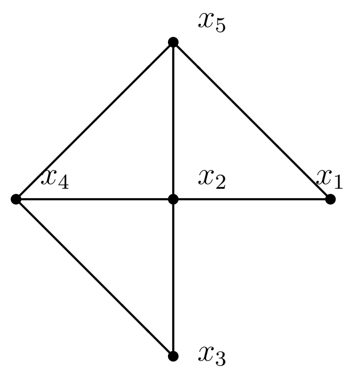


Diagram 13

(d) If $\neg x_2Rx_5$ and $\neg x_1Rx_4$, then we define valuation:

$$V(p) = W \setminus \{x_1\}, \text{ and } V(q) = W \setminus \{x_5\} \text{ and } V(r) = W \setminus \{x_3\}.$$

$$x_3 \models_V \Box p \wedge \Box q, \text{ and } x_3 \not\models_V (\Box p \wedge \Box q) \to r, \text{ and } x_4 \models \Box p, \text{ and } x_4 \not\models_V \Box p \to \Box q.$$

Point x_2 sees x_1, x_3 and x_4 then we get:

$$x_2 \not\models_V \Box p,\ x_2 \not\models_V \Box(\Box p \to \Box q),\ x_2 \not\models_V \Box[(\Box p \wedge \Box q) \to r].$$

Hence: $x_2 \not\models_V (3')$. See Diagram 14.

(2) $\neg x_1Rx_5$.

(a) If x_2Rx_5 and x_1Rx_4, then we define valuation as follows:

$$V(p) = W \setminus \{x_1\} \text{ and } V(q) = W \setminus \{x_3\}, \text{ and } V(r) = W \setminus \{x_5\}.$$

$$x_5 \models_V \Box p \wedge \Box q, \text{ and } x_5 \not\models_V (\Box p \wedge \Box q) \to r, \text{ and } x_3 \models \Box p, \text{ and } x_3 \not\models_V \Box p \to \Box q.$$

Also

$$x_2 \not\models_V \Box p,\ x_2 \not\models_V \Box(\Box p \to \Box q),\ x_2 \not\models_V \Box[(\Box p \wedge \Box q) \to r].$$

Hence: $x_2 \not\models_V (3')$.

(b) If x_2Rx_5 and $\neg x_1Rx_4$, then we define:

$$V(p) = W \setminus \{x_3\} \text{ and } V(q) = W \setminus \{x_5\}, \text{ and } V(r) = W \setminus \{x_1\}.$$

$$x_1 \models_V \Box p \wedge \Box q, \text{ and } x_1 \not\models_V (\Box p \wedge \Box q) \to r, \text{ and } x_5 \models \Box p, \text{ and } x_5 \not\models_V \Box p \to \Box q.$$

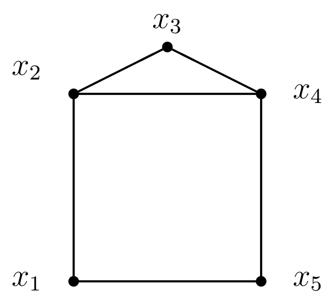


Diagram 14

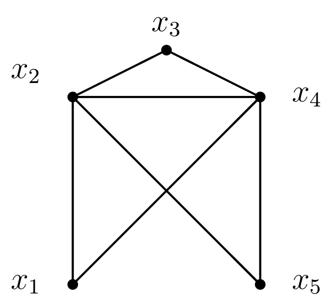


Diagram 15

Also

$$x_2 \not\models_V \Box p,\ x_2 \not\models_V \Box(\Box p \to \Box q),\ x_2 \not\models_V \Box[(\Box p \wedge \Box q) \to r].$$

Hence: $x_2 \not\models_V (3')$. See Diagram 16 below.

(c) If $\neg x_2 R x_5$ and $x_1 R x_4$, then see the above case.

(d) If $\neg x_2 R x_5$ and $\neg x_1 R x_4$, then we define valuation as in (1-d) page 4. See Diagram 17.

$\Box$

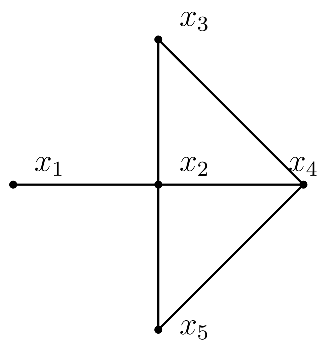


Diagram 16

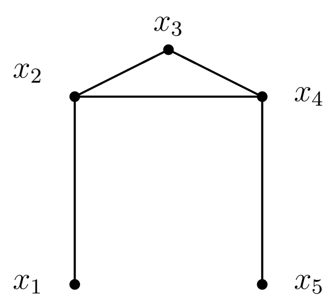


Diagram 17

References

[1] Kostrzycka, Z. (2014) On linear Brouwerian logics, *Mathematical Logic Quarterly* **60** (4-5): 304–313.