On the density of truth in Grzegorczyk's modal logic

Zofia Kostrzycka

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Abstract

The paper is an attempt to count the proportion of tautologies of Grzegorczyk's modal calculus among all formulas. We take advantage of some theorems proved in [2].

1 Introduction

Let L be some logical calculus. Let $|T_n|$ be a number of tautologies of length n of that calculus and $|F_n|$ be a number of all formulas of that length. We define the density $\mu(L)$ as:

$$\mu(L) = \lim_{n\to\infty} \frac{|T_n|}{|F_n|}$$

The number $\mu(L)$ if exists, is an asymptotic probability of finding a tautology among all formulas.
In this paper we continue research concerning the *density of truth* in different logics. Until now, the *density* for both classical and intuitionistic logics of implication of one and two variables are known (see [5],[1]) as well as the *density* of implicational-negational fragments of that logics with one variable (see [8], [3], [4]).
In this note we estimate the *density of truth* for Grzegorczyk's logic and give it exact value for some normal extension of this logic.

2 Grzegorczyk's logic and its normal extensions

Syntactically, Grzegorczyk's logic **Grz** is characterized as a normal extension of **S4** modal calculus by the axiom

$$(grz)\ \ \Box(\Box(p \to \Box p) \to p) \to p$$

The set of rules consists of modus ponens, substitution and necessitation.

The main aim of this paper is to count the *density* of Grzegorczyk's logic. Because of complexity of the problem we have to restrict our investigation to the language $\mathcal{F}^{\{\to,\Box\}}$ consisted of sings of implication and necessity and one propositional variable p only. Its formal definition is including in [2].
We will consider the logics $\mathbf{Grz}^{\leq n} = \mathbf{Grz} \oplus J_n$ (see [2]), containing the logic **Grz** and satisfying the following inclusions:

$$\mathbf{Grz} \subset ... \subset \mathbf{Grz}^{\leq n} \subset \mathbf{Grz}^{\leq n-1} \subset ... \subset \mathbf{Grz}^{\leq 2} \subset \mathbf{Grz}^{\leq 1} \tag{1}$$

3 Counting formulas and generating functions

In this section we set up the way of counting formulas with the established length. We will consider the set $F_n \subseteq \mathcal{F}^{\{\to,\Box\}}$ of all formulas of the length n. The way of measuring the length of formula is set up in [2] [Definition 9].

Definition 1. *By F_n we mean the set of formulas from $\mathcal{F}^{\{\to,\Box\}}$ of the length $n-1$.*

We will also consider some appropriate subclasses of F_n. For example if we have a class $A \in \mathcal{F}^{\{\to,\Box\}}$ then $A_n = F_n \cap A$ and

Definition 2. *By $|A_n|$ we mean the number of formulas from the class A_n.*

Lemma 3. *The number $|F_n|$ of formulas from F_n is given by the recursion:*

$$|F_0| = |F_1| = 0, \ \ |F_2| = 1, \tag{2}$$

$$|F_n| = |F_{n-1}| + \sum_{i=1}^{n-2} |F_i||F_{n-i}|. \tag{3}$$

Proof. Any formula of the length $n-1$ for $n > 2$ is either a necessitation of some formula of the length $n-2$ for which the fragment $|F_{n-1}|$ corresponds, or an implication between some pair of formulas of the lengths $i-1$ and $n-i-1$, respectively. The length of any of such implicational formulas must be $(i-1)+(n-i-1)+1$ which is exactly $n-1$. Therefore the total number of such formulas is $\sum_{i=1}^{n-2} |F_i||F_{n-i}|$. □

The main tool for dealing with asymptotics of sequences of numbers are *generating functions* (see for example [7]). Let $A = (A_0, A_1, A_2, \dots)$ be a sequence of real numbers. It is in one-to-one correspondence to the formal power series $\sum_{n=0}^{\infty} A_n z^n$. Moreover, considering z as a complex variable, this series converges uniformly to a function $f_A(z)$ in some open disc $\{z \in \mathcal{C} : |z| < R\}$. So, with the sequence A we can associate a complex function $f_A(z)$, called the *ordinary generating function* for A, defined in a neighborhood of 0. This correspondence is one-to-one again (unless $R = 0$), since the expansion of a complex function $f(z)$, analytic in a neighborhood of z_0, into a power series $\sum_{n=0}^{\infty} A_n(z - z_0)^n$ is unique, and moreover, this series is the Taylor series, given by

$$A_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0). \tag{4}$$

Many questions concerning the asymptotic behaviour of A can be efficiently resolved by analyzing the behaviour of f_A at the complex circle $|z| = R$.
The key tool will be the following result due to Szegö [6] [Thm. 8.4], see also [7] [Thm. 5.3.2], which relates the generating functions of numerical sequences to the limit of the fractions being investigated. For the technique of proof described below please consult also [5] as well as [8]. We need the following much simpler version of the Szegö lemma.

Lemma 4. *Let $v(z)$ be analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$. If $v(z)$ in the vicinity of $z = 1$ has an expansion of the form*

$$v(z) = \sum_{p \geq 0} v_p (1 - z)^{\frac{p}{2}}, \tag{5}$$

where $p > 0$, and the branch chosen above for the expansion equals $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = v_1 \binom{1/2}{n} (-1)^n + O(n^{-2}). \tag{6}$$

The symbol $[z^n]\{v(z)\}$ stands for the coefficient of z^n in the exponential series expansion of $v(z)$.
First, we determine the generating function for the sequence of numbers $|F_n|$.

Lemma 5. *The generating function f_F for the numbers $|F_n|$ is*

$$f_F(z) = \frac{1 - z}{2} - \frac{\sqrt{(z+1)(1-3z)}}{2}. \tag{7}$$

Proof. The recurrence $|F_n| = |F_{n-1}| + \sum_{i=1}^{n-2} |F_i||F_{n-i}|$ becomes the equality

$$f_F(z) = z f_F(z) + f_F^2(z) + z^2 \tag{8}$$

since the recursion fragment $\sum_{i=1}^{n-2} |F_i||F_{n-i}|$ corresponds exactly with multiplication of power series. The term $|F_{n-1}|$ corresponds with the function $z f_F(z)$.

The quadratic term z^2 corresponds with the first non-zero coefficient in the power series of f_F. Solving the equation we get two possible solutions: $f_F(z) = (1-z)/2 - \sqrt{-3z^2-2z+1}/2$ or $f_F(z) = (1-z)/2 + \sqrt{-3z^2-2z+1}/2$. We have to choose the first one, since it corresponds to the assumption $f_F(0) = 0$ (see equation (2)). $\square$

4 Upper estimation of the *density*

In this section we count the density of the logic $\mathbf{Grz}^{\leq 2}$ (for details see [2]). Since the inclusions (1) hold we conclude that

$$\mu(\mathbf{Grz}) < \mu(\mathbf{Grz}^{\leq n})$$

for every $n \in \mathbb{N}$.
It would be desirable to count the *density* of $\mathbf{Grz}^{\leq n}$ for any $n \in \mathbb{N}$, but we have not been able to do this. Unfortunately, even for $n = 3$ the needed calculations are extremely complicated. We manage to count the density for $n = 2$.
For simplicity of notation we write the quotient algebra $\mathbf{Grz}^{\leq 2}/_{\equiv}$ by AL . It is presented below in the Diagram 1.

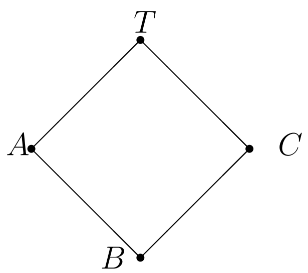

Diagram 1

where

$$A = [p]_{\equiv}, \quad B = [\Box p]_{\equiv}, \quad C = [p \to \Box p]_{\equiv}, \quad T = [p \to p]_{\equiv}$$

Observation 6. *The operations $\{\to, \Box\}$ in the algebra AL can be displayed by the following truth table:*

$\to$	A	B	C	T	$\Box$
A	T	C	C	T	B
B	T	T	T	T	B
C	A	A	T	T	C
T	A	B	C	T	T

Table 1.

For technical reason we are going to consider a new algebra obtained from the one above by an appropriate identification. We take the open filter $[C)$. Let us consider the algebra $AL_1 = AL/_{[C)}$. It is easy to observe that $AL_1 = \mathbf{Grz}^{\leq 1}/_{\equiv}$ and its diagram is the following:

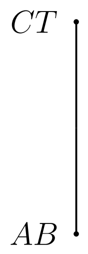

Diagram 2

where

$$AB = A \cup B, \quad CT = C \cup T$$

Observation 7. *The operations $\{\rightarrow, \Box\}$ in the algebra AL_1 are given by the following truth table:*

$\rightarrow$	AB	CT	$\Box$
AB	CT	CT	AB
CT	AB	CT	CT

Table 2.

Now, we determine the generating function f_T for the class T of tautologies of $\mathbf{Grz}^{\leq 2}$. To do that we start with calculating the generating functions f_{AB}, f_{CT} and f_C.

Lemma 8. *The generating function f_{AB} for the numbers $|AB_n|$ is*

$$f_{AB}(z) = \frac{f(z) - 1 + z + X}{2}. \tag{9}$$
$$\textit{where} \quad X = \sqrt{4z^2 + z(f(z) - 2) - f(z) + 1}$$

For simplicity we have written this function in the term of function f. We will repeat it to the other ones.

Proof. Table 2 shows that any formula from the class AB of the length $n-1$ is either a necessitation of formula from the same class AB of the length $n-2$

or an implication of formulas from classes CT and AB of the length $i-1$ and $n-i-1$, respectively. We also know that $p \in AB$. That gives the recurrence

$$|AB_0| = |AB_1| = 0, \quad |AB_2| = 1, \tag{10}$$

$$|AB_n| = |AB_{n-1}| + \sum_{i=1}^{n-1} |CT_i||AB_{n-i}| \tag{11}$$

From disjointness of classes AB and CT we have $|CT_i| = |F_i| - |AB_i|$. Hence $|AB_n| = |AB_{n-1}| + \sum_{i=1}^{n-1}(|F_i| - |AB_i|)|AB_{n-i}|$.
The number $|AB_{n-1}|$ corresponds to the function $zf_{AB}(z)$. The quadratic term z^2 corresponds to the first non-zero coefficient in the power series of f_{AB}. The recursion fragment $\sum_{i=1}^{n-2}(|F_i| - |AB_i|)|AB_{n-i}|$ corresponds exactly to multiplication of power series. Hence we have the equation:

$$f_{AB}(z) = (f(z) - f_{AB}(z))f_{AB}(z) + zf_{AB}(z) + z^2. \tag{12}$$

By solving it with the boundary condition $f_{AB}(0) = 0$ we have (9).

□

Corollary 9. *The generating function f_{CT} for the numbers $|CT_n|$ is*

$$f_{CT}(z) = \frac{f(z) + 1 - z - X}{2}. \tag{13}$$
$$\textit{where} \quad X = \sqrt{4z^2 + z(f(z) - 2) - f(z) + 1}$$

Proof. It follows from disjointness of classes AB and CT that $f_{CT} = f - f_{AB}$.

□

Lemma 10. *The generating function f_C for the numbers $|C_n|$ is*

$$f_C(z) = \frac{1}{6}\left(2^{\frac{2}{3}}Y - \frac{2^{\frac{4}{3}}U}{Y} - X - z + 3(f-1)\right) \tag{14}$$

where

$$Y = \sqrt[3]{S + \sqrt{4U^3 + S^2}},$$
$$S = \frac{1}{2}\left(X(19z^2 + 2z(11f - 13) - 4(f-1)) + 43z^3 + 3z^2(7f - 17) + 30z(1-f)\right),$$
$$U = -\frac{1}{2}\left(zX + z^2 - z(f+1) - 2(f-1)\right),$$
$$X = \sqrt{4z^2 + z(f-2) - f + 1}$$

For simplicity we have omitted in the above function the argument (z) and have written f instead of $f(z)$. We will repeat it hereafter.

Proof. From Table 1 we can notice the following recurrence for the numbers $|B_n|$ holds:

$$\begin{aligned} |B_0| &= 0, \quad |B_1| = 0 \\ |B_n| &= (|A_{n-1}| + |B_{n-1}|) + \sum_{i=1}^{n-1} |T_i||B_{n-i}| \end{aligned} \tag{15}$$

This can be translated into equation:

$$f_B = f_T f_B + (f_A + f_B)z. \tag{16}$$

Since $f_A + f_B = f_{AB}$ and $f_T = f_{CT} - f_C$ then we have:

$$f_B = \frac{z f_{AB}}{1 - f_{CT} + f_C}. \tag{17}$$

Table 1 suggests also that the recursion schema for the class C must be:

$$\begin{aligned} |C_0| &= 0, \quad |C_1| = 0 \\ |C_n| &= |C_{n-1}| + \sum_{i=1}^{n-1} (|A_i|(|B_{n-i}| + |C_{n-i}|) + |T_i||C_{n-i}|) \end{aligned} \tag{18}$$

The above recurrence gives the following equality between generating functions:

$$f_C = z f_C + (f_B + f_C) f_A + f_T f_C \tag{19}$$

The unknown functions from (19) can be replaced by the already known. We know that $f_A = f_{AB} - f_B$ and $f_T = f_{CT} - f_C$. After application of the above equalities to the (19) we get

$$f_C = z f_C + ((f_{AB} - f_A + f_C)(f_{AB} - f_B) + (f_{CT} - f_C) f_C \tag{20}$$

From the system of equations

$$\begin{cases} (17) \\ (20) \end{cases}$$

we obtained a four-degree equation with the unknown function f_C. To solve it we had to intensively use *Mathematica* package and from four solutions we chose one satisfying the boundary condition $f_C(0) = 0$. Then we have (14) presenting the function f_C in terms of some expressions Y, S, U, X. □

Corollary 11. *The generating function f_T for the numbers $|T_n|$ is*

$$f_T = f_{CT} - f_C \tag{21}$$

where the functions f_{CT} and f_C are defined by (13) and (14).

To apply the Szegö lemma we have to have functions which are analytic in the open disc $|z| < 1$, and the nearest singularity is at $z_0 = 1$. For that purpose we are going to calibrate functions f and f_T in the following way:

$$\begin{aligned} \widehat{f}(z) &= f\left(\tfrac{z}{3}\right) & \widehat{f_{CT}}(z) &= f_{CT}\left(\tfrac{z}{3}\right) \\ \widehat{f_C}(z) &= f_C\left(\tfrac{z}{3}\right) & \widehat{f_T}(z) &= f_T\left(\tfrac{z}{3}\right). \end{aligned}$$

After appropriate simplification of the above expressions we get the following:

$$\widehat{f}(z) = \frac{1}{6}\left(3 - z - \sqrt{3}\sqrt{(z+3)(1-z)}\right) \tag{22}$$

$$\widehat{f_{CT}}(z) = \frac{3\widehat{f} + 3 - z - \widehat{X}}{6} \tag{23}$$

$$\widehat{f_C}(z) = \frac{(2^{\frac{2}{3}}\widehat{Y} - \frac{2^{\frac{4}{3}}\widehat{U}}{\widehat{Y}} - \widehat{X} - z + 9(\widehat{f} - 1)}{18} \tag{24}$$

$$\widehat{f_T} = \widehat{f_{CT}} - \widehat{f_C} \tag{25}$$

where

$$\begin{aligned} \widehat{Y} &= \sqrt[3]{\widehat{S} + \sqrt{4\widehat{U}^3 + \widehat{S}^2}}, \\ \widehat{S} &= \frac{1}{54}\Big(3\widehat{X}(19z^2 + 6z(11\widehat{f} - 13) - 36(\widehat{f} - 1)) + 43z^3 + \\ &\quad 9z^2(7\widehat{f} - 17) + 270z(1 - \widehat{f})\Big), \\ \widehat{U} &= -\frac{1}{18}\left(3z\widehat{X} + z^2 - 3z(\widehat{f} + 1) - 18(\widehat{f} - 1)\right), \\ \widehat{X} &= \frac{1}{3}\sqrt{4z^2 + 3z(\widehat{f} - 2) - 9\widehat{f} + 9} \end{aligned}$$

Note that relations between power series of appropriate functions are such as $[z^n]\{f(z)\} = \left([z^n]\{\widehat{f}(z)\}\right) 3^n$.

Lemma 12. *$z_0 = 1$ is the only singularity of $\widehat{f}$ and $\widehat{f_T}$ located in $|z| \leq 1$.*

Proof. It is easy to observe the function $\widehat{f}(z)$ has only singularities at $z = 1$ and $z = -3$. To make sure the function $\widehat{f_T}(z)$ has the nearest one at $z = 1$, we had to solve the following complicated equations:

$$\begin{aligned} \widehat{X} &= 0 \\ \widehat{Y} &= 0 \\ 4\widehat{U}^3 + \widehat{S}^2 &= 0 \end{aligned}$$

To do that we had to extensively use *the Mathematica* package and it occurred that all solutions which are different from $z = 1$ are situated outside the disc $|z| \leq 1$.

Theorem 13. *Expansions of functions $\widehat{f}$ and $\widehat{f_T}$ in a neighborhood of $z = 1$ are as follows:*

$$\widehat{f}(z) = f_0 + f_1\sqrt{1-z} + ...$$
$$\widehat{f_T}(z) = t_0 + t_1\sqrt{1-z} + ...$$

where

$$f_0 = \frac{1}{3}, \quad f_1 = -\frac{1}{\sqrt{3}}, \quad \ldots, \quad t_0 = 0.104415..., \quad t_1 \quad = \quad -0.356051...$$

Proof. The above coefficients have been found using *the Mathematica* package. The exact values of the coefficients t_0 and t_1 are too long to be written here. □

Now, we can calculate the density of implicational-necessitional part of extension of Grzegorczyk's logic $\mathbf{Grz}^{\leq 2}$ of one variable. By applying the Szegö lemma we get as follows:

Theorem 14.

$$\begin{aligned}\mu(\mathbf{Grz}^{\leq 2}) &= \lim_{n\to\infty} \frac{|T_n|}{|F_n|} = \lim_{n\to\infty} \frac{(t_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n} \\ &= \lim_{n\to\infty} \frac{t_1}{f_1}(1+o(1)) = \frac{t_1}{f_1} \approx 61.27\%\end{aligned}$$

5 Lower estimation of the *density*

Definition 15. *The set of simple modal tautologies is defined as follows:*

1. $p \to p \in ST$,
2. $\Box(\Box(p \to \Box p) \to p) \to p \in ST$,
3. *If* $\alpha \in ST$ *then* $\Box\alpha \in ST$,
4. *If* $\alpha \in ST$ *then* $\beta \to \alpha \in ST$ *for every* $\beta \in \mathcal{F}^{\{\to,\Box\}}$,
5. *If* $\alpha \notin ST$, *then* $\underbrace{\Box ... \Box}_{k-times} p \to \alpha \in ST$ *for* $k \geq 1$.

From the above definition it is easy to notice the set of simple tautologies is a proper subset of the set of the ones of Grzegorczyk's logic. Hence we have:

Observation 16. $\mu(ST) < \mu(\mathbf{Grz})$

Lemma 17. *The numbers $|ST_n|$ of formulas from ST_n are given by the recursion:*

$$|ST_0| = ... = |ST_3| = 0, \quad |ST_4 = 1|, \tag{26}$$

$$|ST_n| = |ST_{n-1}| + \sum_{i=1}^{n-2} |F_{n-i}||ST_i| + \underbrace{((|F_{n-3}| - |ST_{n-3}|) + (|F_{n-4}| - |ST_{n-4}|) + ... + (|F_2| - |ST_2|))}_{(n-4)-times} \tag{27}$$

Proof. From Definition 15 we see the simple modal tautologies of the length $n-1$ are either a necessitation of simple modal tautology of the length $n-2$ or an implication of some pairs consisted of any formula and a simple modal tautology or the formula $\underbrace{\Box...\Box}_{k-times} p$ and any formula which is not a simple modal tautology. □

Lemma 18. *The generating function f_{ST} for the numbers $|ST_n|$ is the following:*

$$f_{ST}(z) = \frac{z^4 + z^{11} + \frac{fz^3(1-z^{-4+n})}{1-z}}{1 - f - z + \frac{z^3(1-z^{-4+n})}{1-z}} \tag{28}$$

Proof. From the recurrence (27) we obtain the generating function f_{ST} must satisfy the following equation:

$$f_{ST}(z) = f_{ST}(z)z + f(z)f_{ST}(z) + (f(z) - f_{ST}(z))(z^3 + z^4 + ... + z^{n-2}) + z^4 + z^{11} \tag{29}$$

Since $z^3 + z^4 + ... + z^{n-2} = z^3 \frac{1-z^{n-4}}{1-z}$ then after solving (29) with the boundary condition $f_{ST}(0) = 0$ we get (28). □

Analogously as in the previous section we calibrate the function f_{ST}:

Definition 19. $\widehat{f_{ST}}(z) = f_{ST}(\frac{z}{3})$.

After a suitable substitution we have:

$$\widehat{f_{ST}}(z) = \frac{\left(\frac{z}{3}\right)^4 + \left(\frac{z}{3}\right)^{11} + \frac{fz^3(1-3^{4-n}z^{-4+n})}{9(3-z)}}{1 - f - \frac{z}{3} + \frac{z^3(1-3^{4-n}z^{-4+n})}{9(3-z)}} \tag{30}$$

Now, we should check that the only singularity is situated in disc $|z| \leq 1$ of the function (28) is the point $z = 1$. We set up that $n = 10$ (which has no significant influence for our calculations) and obtain:

$$\widehat{f_{ST}}^*(z) = \frac{\left(\frac{z}{3}\right)^4 + \left(\frac{z}{3}\right)^{11} + \frac{fz^3(1-3^{-6}z^6)}{9(3-z)}}{1 - f - \frac{z}{3} + \frac{z^3(1-3^{-6}z^6)}{9(3-z)}} \tag{31}$$

Lemma 20. $z_0 = 1$ *is the only singularity of the function* $\widehat{f_{ST}}^*$ *located in* $|z| \leq 1$.

Proof. We check that the following equation has no solution at the disc $|z| \leq 1$:

$$1 - f - \frac{z}{3} + \frac{z^3(1 - 3^{-6}z^6)}{9(3 - z)} \quad = \quad 0$$

We used *the Mathematica* package. □

Theorem 21. *Expansion of function* $\widehat{f_{ST}}^*$ *in a neighborhood of* $z = 1$ *is as follows:*

$$\widehat{f_{ST}}^*(z) = t_0^* + t_1^*\sqrt{1 - z} + \ldots$$

where

$$t_0^* = \frac{5464}{68877}, \quad t_1^* = -\frac{2256316\sqrt{3}}{19522803}, \ldots$$

Proof. The above coefficients have been found using *the Mathematica* package. □

Now, we have the value of the *density* of the set of simple modal tautologies:

Theorem 22.

$$\begin{aligned} \mu(ST) &= \lim_{n\to\infty} \frac{|ST_n|}{|F_n|} = \lim_{n\to\infty} \frac{(t_1^*\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n} \\ &= \lim_{n\to\infty} \frac{t_1^*}{f_1}(1 + o(1)) = \frac{t_1^*}{f_1} \approx 34.67\% \end{aligned}$$

Theorems 14 and 22 give us some information about the *density* of implicational-necessitational fragment of Grzegorczyk's logic of one variable. We know only that:

$$34.67\% < \mu(\mathbf{Grz}) < 61.27\% \qquad (32)$$

Since the method of counting the *densities* of $\mathbf{Grz}^{\leq 3}$ is the same as the one of $\mathbf{Grz}^{\leq 2}$ (see Diagram 3 in [2]), we hope the inequalities will be soon improved, especially, the upper estimation. The only problem in that case are degrees of complexity of some equations.

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Politechnika Opolska
Luboszycka 3, 45-036 Opole, Poland
E-mail `zkostrz@polo.po.opole.pl`