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Functional completeness and density of truth of logics

Zofia Kostrzycka
University of Technology,
Opole, Poland

$A \subset Form, \quad ||.|| : Form \to N$

Definition 1. *We associate the density* $\mu(A)$ *with a subset* A *of formulas as:*

$$\mu(A) = \lim_{n\to\infty} \frac{\#\{t \in A : \|t\| = n\}}{\#\{t \in Form : \|t\| = n\}} \tag{1}$$

if the appropriate limit exists.

Densities of some fragments of classical and intuitionistic

$$\mu(Cl_p^{\rightarrow}) = \mu(Int_p^{\rightarrow}) \approx 72.36\%$$

$$\mu(Cl_{p,q}^{\rightarrow}) \approx 51.9\%$$
$$\mu(Int_{p,q}^{\rightarrow}) \approx 50.43\%$$

[1] Moczurad M., Tyszkiewicz J., Zaionc M. *Statistical properties of simple types*, Mathematical Structures in Computer Science, vol 10, 2000, pp 575-594.

[2] Kostrzycka Z., *On the density of implicational parts of intuitionistic and classical logics*, Journal of Applied Non-Classical Logics, Vol. 13, Number 3, 2003, pp 295-325.

$$\mu(Cl_p^{\rightarrow,\neg}) \approx 42.3\%$$
$$\mu(Int_p^{\rightarrow,\neg}) \approx 39.5\%$$

$$\mu(CL_{p,\neg p}^{\wedge,\vee}) \approx 28.8\%$$
$$\mu(CL_{p,q,\neg p,\neg q}^{\wedge,\vee}) \approx 20.9\%$$

[3] Z. Kostrzycka, M. Zaionc, *Statistics of intuitionistic versus classical logics*, Studia Logica, Vol. 76, Number 3, 2004, pp 307 - 328.

[4] D. Gardy and A.R. Woods, *And/or tree probabilities of Boolean functions*, Discrete Mathematics and Theoretical Computer Science, 2005, pp 139-146.

Questions:

1. When does the density of truth exist and how it depends on the chosen language?

2. If the density of truth depends on the chosen language, which connectives are preserving truth, which are not?

Definability in propositional logic

In logic, a set of logical connectives is functionally complete if all other possible connectives can be defined in terms of it.

In algebra, the notion of functional completeness is similar. Let $A_n = \{1, 2, 3..., n\}$ and $\mathbf{A_n} = (A_n, f_1, ..., f_n)$.

The algebra $\mathbf{A_n}$ is functionally complete if any function $f : A_n^k \to A_n$ ($k \geq 0$) can be represented as a superposition of the functions $f_1, ..., f_n$.

Example:

The algebra of classical logic $\mathbf{B_2} = (\{0, 1\}, \neg, \rightarrow, \vee, \wedge, \equiv)$ is functionally complete.

Theorem 2. *[Post] If the algebra A_n is functionally complete for the m-variables connectives ($m \geq 2$), then it is functionally complete for the $m+1$-variables connectives and so functionally complete.*

The above theorem reduces the notion of functional completeness of algebras of logics to the problem of definability of all binary and unary logical connectives.

The classical logic is characterized by two-valued matrix. That gives us that every classical formula containing n propositional variables, generates a corresponding truth function of n- arguments. In the set $Form$ we can introduce an equivalence relation $\equiv$ in the conventional way:

$$\alpha \equiv \beta \text{ iff } \forall_{h:Var \to \{0,1\}} h(\alpha) = h(\beta)$$

The equivalence relation $\equiv$ is also a congruence relation and the quotient algebra $Form/_{\equiv}$ is called the Lindembaum algebra of the logic CL. We denote it by:

$$AL(CL) = Form/_{\equiv} \tag{2}$$

Theorem 3. *The Lindembaum algebra $AL(CL(1))$ consists of 4 equivalence classes and is isomorphic to the Boolean algebra $B_4 = B_2 \times B_2$.*

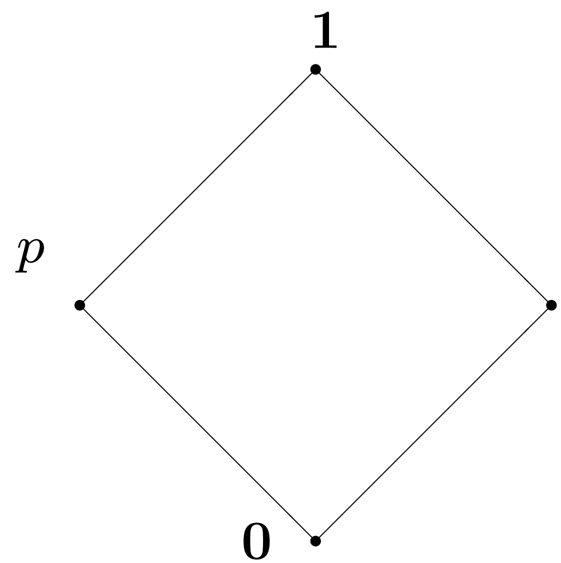
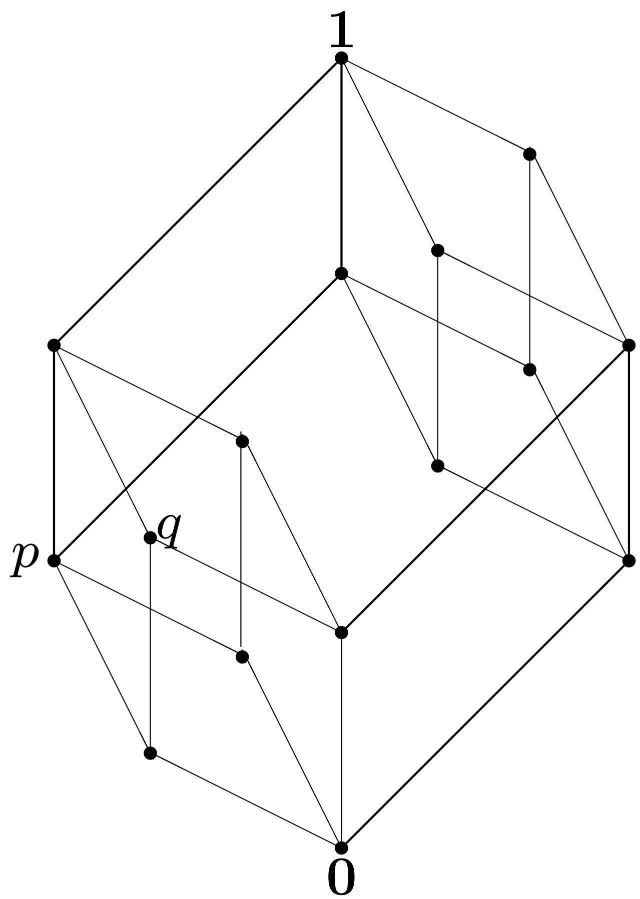


Diagram 1

Theorem 4. *The Lindembaum algebra $AL(CL(2))$ consists of 16 equivalence classes and is isomorphic to the Boolean algebra $B_{16} = B_4 \times B_4$.*



Corollary 5. *Let* $Cl_1^{\{f_i, 1 \leq i \leq n\}}$ *be some functionally complete fragment of classical logic with one variable. Then the appropriate Lindenbaum algebra consists of 4 classes.*

Corollary 6. *Let* $Cl_2^{\{f_i, 1 \leq i \leq n\}}$ *be some functionally complete fragment of classical logic with two variables. Then the appropriate Lindenbaum algebra consists of 16 classes.*

Definition 7. *We associate the density $\mu(\mathcal{A})$ with a subset $\mathcal{A}$ of formulas as:*

$$\mu(\mathcal{A}) = \lim_{n\to\infty} \frac{\#\{t \in \mathcal{A} : \|t\| = n\}}{\#\{t \in Form : \|t\| = n\}} \tag{3}$$

if the appropriate limit exists.

The Drmota-Lalley-Woods theorem

Theorem 8. *Consider a nonlinear polynomial system, defined by a set of equations*

$$\{\overrightarrow{y} = \Phi_j(z, y_1, ..., y_m)\}, \quad 1 \leq j \leq m$$

which is a-proper, a-positive, a-irreducible and a-aperiodic. Then

1. *All component solutions y_i have the same radius of convergence $\rho < \infty$.*

2. *There exist functions h_j analytic at the origin such that*

$$y_j = h_j(\sqrt{1 - z/\rho}), \quad (z \to \rho^-). \tag{4}$$

3. *All other dominant singularities are of the form $\rho\omega$ with ω being a root of unity.*

4. *If the system is a-aperiodic then all y_j have ρ as unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form:*

$$[z^n]y_j(z) \sim \rho^{-n} \left(\sum_{k \geq 1} d_k n^{-1-k/2} \right). \tag{5}$$

Application of the Drmota-Lalley-Woods theorem

Suppose we have two functions f_T and f_F enumerating the tautologies of some logic and all formulas. Suppose they have the same dominant singularity ρ and there are the suitable constants α_1, α_2, β_1, β_2 such that:

$$f_T(z) = \alpha_1 - \beta_1\sqrt{1 - z/\rho} + O(1 - z/\rho), \tag{6}$$

$$f_F(z) = \alpha_2 - \beta_2\sqrt{1 - z/\rho} + O(1 - z/\rho). \tag{7}$$

Then the *density of truth* (probability that a random formula is a tautology) is given by:

$$\mu(T) = \lim_{n\to\infty} \frac{[z^n]f_T(z)}{[z^n]f_F(z)} = \frac{\beta_1}{\beta_2}. \tag{8}$$

Definition 9. *The set of formulas* $Form_k^{f_1}$ *over* k *propositional variables is a minimal set consisting of these variables and closed under some connective* $\circ$. *In this definition the norm* $\|.\|$ *measures the total number of appearances of propositional variables in the formula. The set of formulas of length* n *is denoted by* F_n^k.

Lemma 10. *The numbers* $|F_n^k|$ *are given by the following recursion:*

$$|F_0^k| = 0, \ |F_1^k| = k, \tag{9}$$

$$|F_n^k| = \sum_{i=1}^{n-1} |F_i^k||F_{n-i}^k| \tag{10}$$

Lemma 11. *The generating function f_{F^k} for the numbers F_n^k is the following:*

$$f_{F^k}(z) = \frac{1 - \sqrt{1 - 4kz}}{2} \tag{11}$$

Proof. From recurrence (10) we see that the generating function f_{F^k} has to fulfil the following equation:

$$f_{F^k}(z) = f^2_{F^k}(z) + kz \tag{12}$$

System $(CL_1, /)$

Lemma 12. *The Lindenbaum algebra $AL(Cl_1^{/})$ for $\{/\}$-fragment of classical logic of one variable consists of the following four classes:*

$$
\begin{aligned}
A &= [p]_{\equiv}, \\
B &= [p/p]_{\equiv}, \\
N &= [(p/p)/p]_{\equiv}, \\
T &= [((p/p)/p)/((p/p)/p)]_{\equiv}.
\end{aligned}
$$

$/$	N	A	B	T
N	T	B	A	N
A	B	B	N	N
B	A	N	A	N
T	N	N	N	N

$$
\begin{aligned}
f_T(z) &= f_N(z) f_N(z), && (13)\\
f_A(z) &= 2 f_B(z) f_N(z) + f_B(z) f_B(z) + z, && (14)\\
f_B(z) &= 2 f_A(z) f_N(z) + f_A(z) f_A(z), && (15)\\
f_N(z) &= 2 f_{F^1}(z) f_T(z) - f_T^2, && (16)
\end{aligned}
$$

The generating functions for classes of tautologies and non-tautologies f_T, f_N, are as follows:

$$
\begin{aligned}
f_T(z) &= \frac{1}{2}\Bigg(3 + 2f - \sqrt{1+4f} - \sqrt{1+4f-4z} \\
&\quad - \sqrt{5 + 4f - 2\sqrt{1+4f} - 2\sqrt{1+4f-4z}}\Bigg) \\
f_N(z) &= \frac{1}{2}\left(1 - \sqrt{5 + 4f - 2\sqrt{1+4f} - 2\sqrt{1+4f-4z}}\right)
\end{aligned}
$$

Expansions of f_{F^1} and f_T around $z_0 = 1/4$:

$$
\begin{aligned}
f_N(z) &= \alpha + \beta\sqrt{1-4z} + O(1-4z), \\
f_T(z) &= \gamma + \delta\sqrt{1-4z} + O(1-4z), \\
f_{F^1}(z) &= \frac{1}{2} - 2\sqrt{1-4z} + O(1-4z),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \frac{1}{2}\left(1 - \sqrt{7 - 2\sqrt{2} - 2\sqrt{3}}\right) \\
\beta &= -\frac{-2 + \sqrt{2} + \frac{2}{\sqrt{3}}}{\sqrt{7 - 2\sqrt{2} - 2\sqrt{3}}} \\
\gamma &= \frac{1}{2}\left(4 - \sqrt{2} - \sqrt{3} - \sqrt{7 - 2\sqrt{2} - 2\sqrt{3}}\right) \\
\delta &= 2\left(-1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{-2 + \sqrt{2} + \frac{2}{\sqrt{3}}}{2\sqrt{7 - 2\sqrt{2} - 2\sqrt{3}}}\right)
\end{aligned}
$$

Theorem 13. *The density of the class* $Cl_1^{/}$ *is the following:*

$$\mu(Cl_1^{/}) \approx 5.4\%.$$

Analogously we may count the density of the class of non-tautologies $NCl_1^{/}$:

$$\mu(NCl_1^{/}) \approx 33.8\%$$

System Cl_2'

Theorem 14. *For $k = 2$ the asymptotic density of the set of classical tautologies Cl_2' exists and is the following:*

$$\mu(Cl_2') \approx 3.1\% \ .$$

Analogously we may count the density of the class of non-tautologies NCl_2':

$$\mu(NCl_2') \approx 18.7\%.$$

System $Cl^{|}_{1}$

Translation f from $(CL, /)$ onto $(CL, |)$:

$$\begin{aligned} f(p_i) &= p_i \\ f(\alpha/\beta) &= f(\alpha)|f(\beta) \end{aligned}$$

Observation 15. *For any* $\alpha \in Form$

$$\begin{aligned} \alpha \in Cl^{/} \quad &\textit{iff} \quad f(\alpha) \in NCl^{|} \\ \alpha \in Cl^{|} \quad &\textit{iff} \quad f(\alpha) \in NCl^{/} \end{aligned}$$

Corollary 16.

$$\mu(Cl^{|}_{1}) \approx 33.8\%$$

$$\mu(NCl^{|}_{1}) \approx 5.4\%$$

$$\mu(Cl^{|}_{2}) \approx 18.7\%$$

$$\mu(NCl^{|}_{2}) \approx 3.1\%$$

Functional completeness and asymptotic density of logics with one binary and one unary connectives

Definition 17. *The set* $Form_k^{\{f_1,f_2\}}$ *over* k *propositional variables is a minimal set consisting of these variables and closed under* f_1, f_2*. In this definition the norm* $\|\phi\|$ *means the total number of characters in formula* ϕ *without parentheses.*

System $Cl_1^{\wedge,\neg}$

Theorem 18. *For $k = 1$ asymptotic density of the set of classical tautologies $Cl_1^{\wedge,\neg}$ exists and is:*

$$\mu(Cl_1^{\wedge,\neg}) \approx 19,36\%$$

Also:

$$\mu(NCl_1^{\wedge,\neg}) \approx 55,13\%$$

System $Cl_2^{\wedge,\neg}$

Theorem 19. *For $k = 2$ asymptotic density of the set of classical tautologies $Cl_2^{\wedge,\neg}$ exists and is:*

$$\mu(Cl_2^{\wedge,\neg}) \approx 15.14\%$$

And

$$\mu(NCl_2^{\wedge,\neg}) \approx 55.87\%$$

Systems $Cl_1^{\vee,\neg}$ and $Cl_2^{\vee,\neg}$

Translation f from $(CL, \wedge, \neg)$ onto $(CL, \vee, \neg)$:

$$\begin{aligned} f(p_i) &= p_i \\ f(\alpha \wedge \beta) &= f(\alpha) \vee f(\beta) \\ f(\neg\alpha) &= \neg f(\alpha) \end{aligned}$$

We have the following observation:

Observation 20. *For any* $\alpha \in Form$

$$\alpha \in Cl^{\wedge,\neg} \quad \textit{iff} \quad f(\alpha) \in NCl^{\vee,\neg}$$
$$\alpha \in Cl^{\vee,\neg} \quad \textit{iff} \quad f(\alpha) \in NCl^{\wedge,\neg}$$

Corollary 21.

$$\mu(Cl_1^{\vee,\neg}) \approx 55.13\%$$
$$\mu(NCl_1^{\vee,\neg}) \approx 19.36\%$$
$$\mu(Cl_2^{\vee,\neg}) \approx 55.87\%$$
$$\mu(NCl_2^{\vee,\neg}) \approx 15.14\%$$

Conjecture 22.

$$\mu(Cl_k^{\vee,\neg}) \leq \mu(Cl_{k+1}^{\vee,\neg})$$

Conjecture 23.

$$\lim_{k\to\infty} \mu(Cl_k^{\vee,\neg}) \neq 0$$

Systems $Cl_1^{\to,\neg}$ and $Cl_2^{\to,\neg}$

$$\mu(Cl_1^{\to,\neg}) \approx 0.4232...^{*} .$$

$$\mu(NCl_1^{\to,\neg}) \approx 16.3\%$$

$$\mu(Cl_2^{\to,\neg}) \approx 33.1\%$$
$$\mu(NCl_2^{\to,\neg}) \approx 9.71\%$$

* Zaionc, M. *On the asymptotic density of tautologies in logic of implication and negation*, Reports on Mathematical Logic, vol. 39, 2004.

Systems $Cl^{\rightarrow}_{p,\mathbf{0}}$ and $Cl^{\rightarrow}_{p,q,\mathbf{0}}$

$$\neg\alpha = \alpha \rightarrow \mathbf{0}$$

Examples of tautologies:

$\alpha \rightarrow \neg\neg\alpha \;=\; \alpha \rightarrow ((\alpha \rightarrow \mathbf{0}) \rightarrow \mathbf{0})$

$\mathbf{0} \rightarrow \alpha$

$\mu(Cl^{\rightarrow}_{p,\mathbf{0}}) \approx 62.1\%$
$\mu(NCl^{\rightarrow}_{p,\mathbf{0}}) \approx 10.9\%$
$\mu(Cl^{\rightarrow}_{p,q,\mathbf{0}}) \approx 51.49\%$
$\mu(NCl^{\rightarrow}_{p,q,\mathbf{0}}) \approx 5.6\%$

Densities of logics with equivalence

In [5] and [6] it is proved that the density of truth for the logics: $Cl_p^{\leftrightarrow}$ and $Cl_{p,g}^{\leftrightarrow}$ do not exist. The negative result is strongly connected with existing not one, but two singularities laying on the same circle.

[5] Matecki G. Asymptotic density for equivalence, *Electronic Notes in Theoretical Computer Science URL*,140:81-91, 2005.

[6] Kostrzycka Z., *On asymptotic divergency in equivalential logics*, accepted to MSCS.

The generating function for the class of tautologies: $Cl_1^{\leftrightarrow}$ is as follows:

$$f_{T^1}(z) = \frac{1}{4}\left(2 - \sqrt{1-4z} - \sqrt{1+4z}\right).$$

It has two singularities and $z_1 = \frac{1}{4}$ and $z_2 = -\frac{1}{4}$. It is because the system of equation of the appropriate generating functions is not a-aperiodic.

A cursory analysis gives us the expansion of generating function for tautologies $Cl_1^{\leftrightarrow}$:

$$f_{T^1}(z) = 1z^2 + 5z^4 + 42z^6 + ...$$

Of course the $\{\leftrightarrow\}$ language is not the functionally complete one. The Lindenbaum algebras for $Cl_p^{\leftrightarrow}$ and $Cl_{p,g}^{\leftrightarrow}$ look as follows:

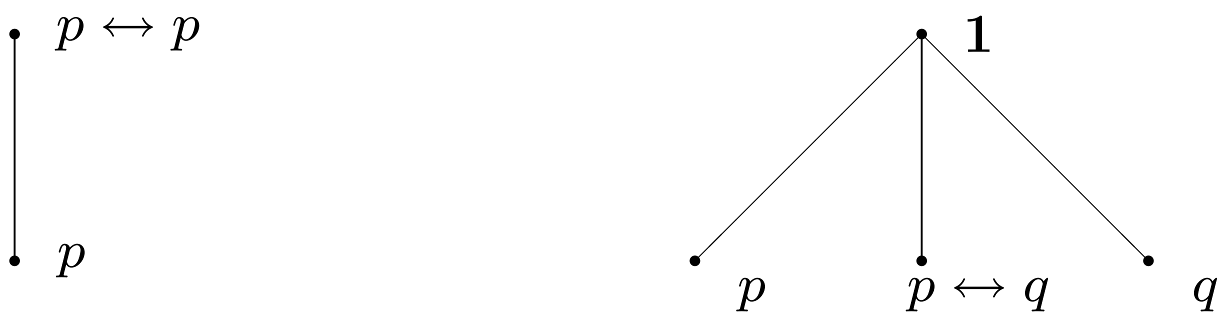


Diagram 3.

Systems $Cl_p^{\leftrightarrow,\neg}$ and $Cl_{p,g}^{\leftrightarrow,\neg}$

The set of connectives is not functionally complete, but in the case of one variable the Lindenbaum algebra is a four-element one:

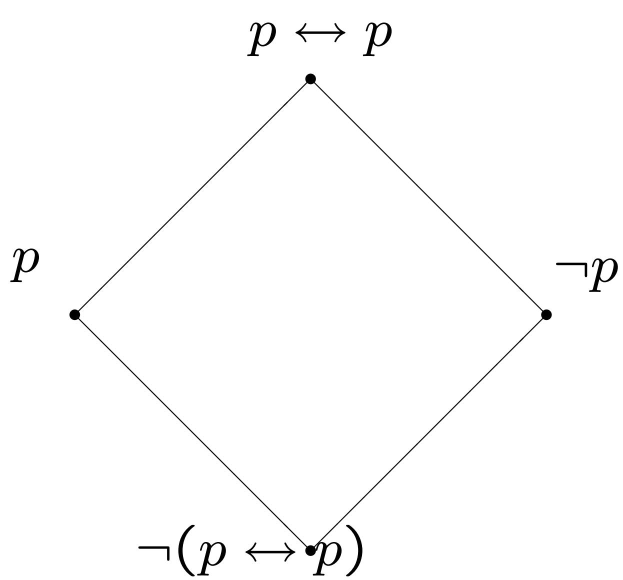
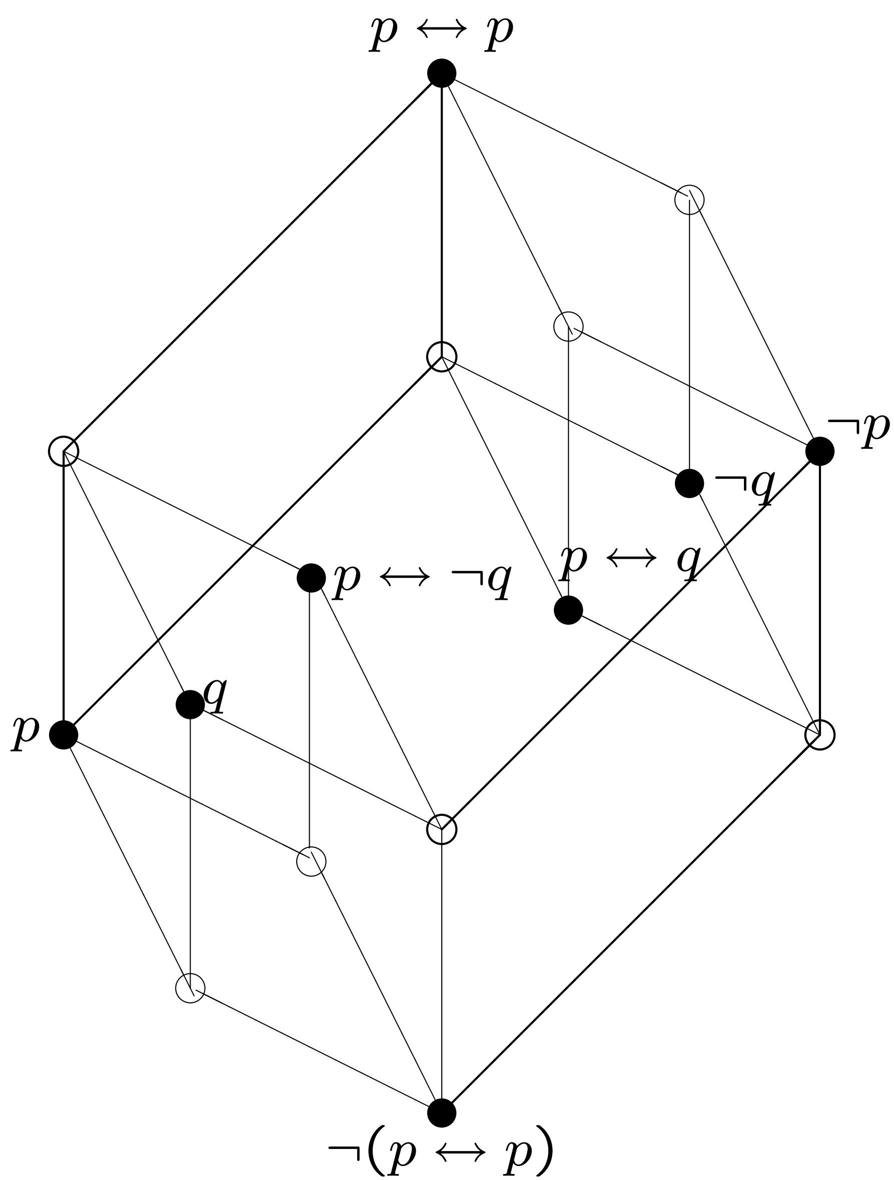


Diagram 4

In the case of two variables we obtain eight-element subalgebra of the Boolean algebra B_{16}.



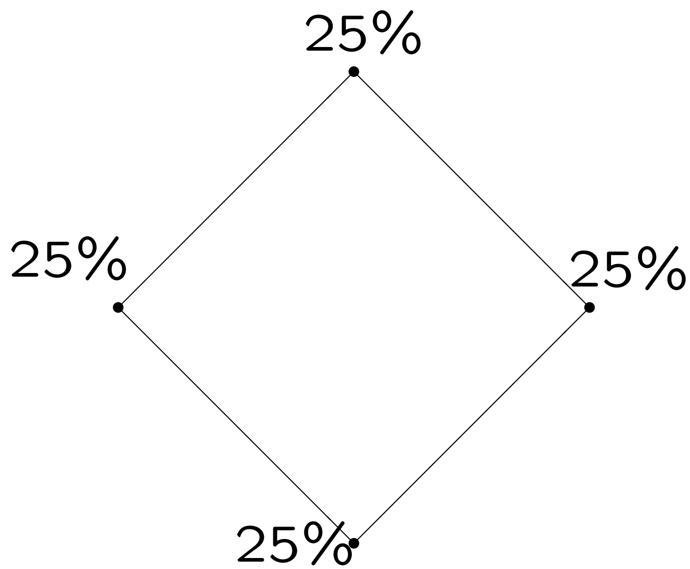
$p \leftrightarrow p$
$\neg p$
$\neg q$
$p \leftrightarrow \neg q$
$p \leftrightarrow q$
q
p
$\neg(p \leftrightarrow p)$

The densities of truth for the logics: $Cl_p^{\leftrightarrow,\neg}$ and $Cl_{p,g}^{\leftrightarrow,\neg}$ do not exist.

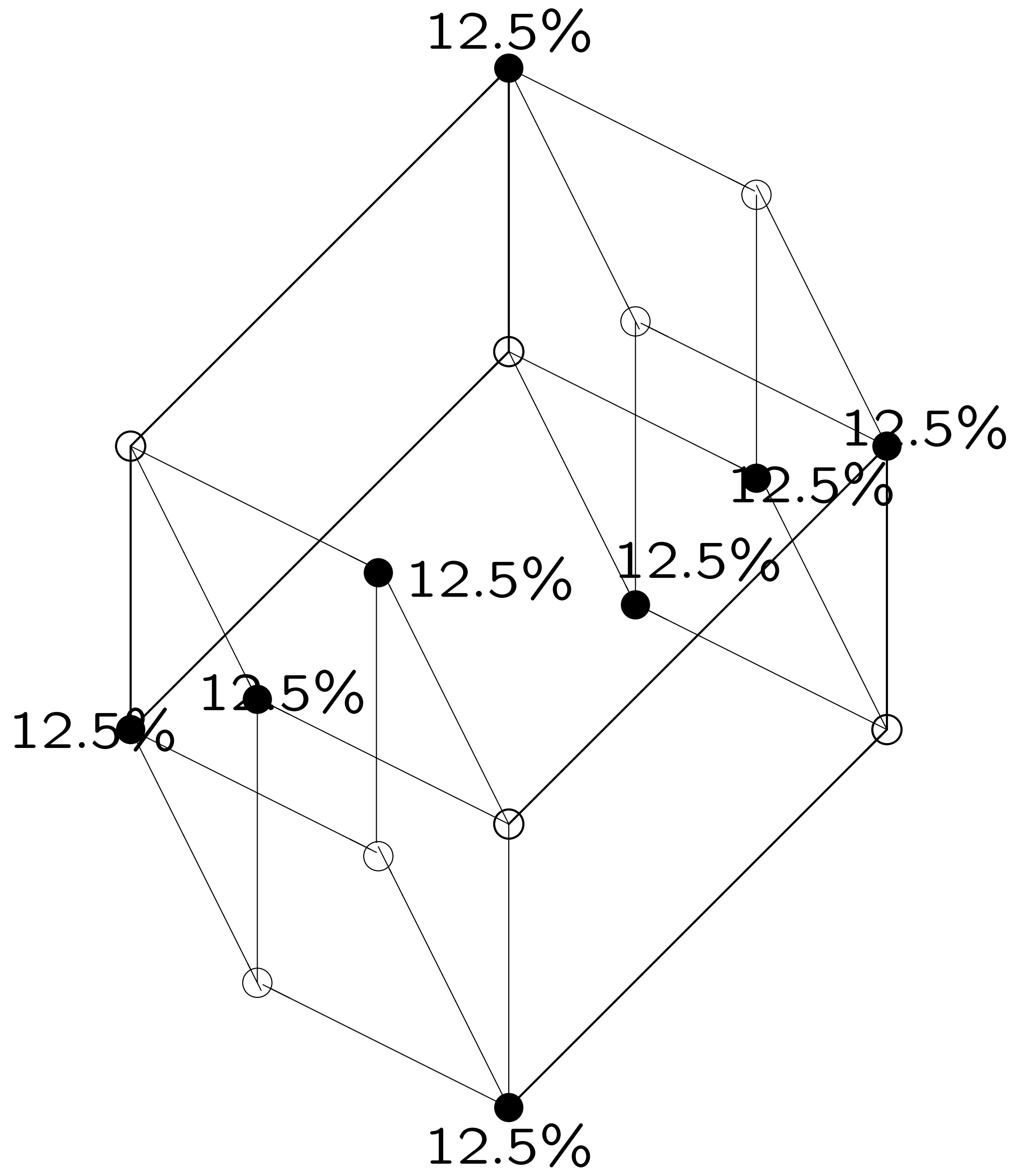
The same situations holds if we consider $Cl_{p,\mathbf{0}}^{\leftrightarrow}$ and $Cl_{p,g,\mathbf{0}}^{\leftrightarrow}$.

Systems $Cl^{\leftrightarrow}_{p,\mathbf{0},\mathbf{1}}$ and $Cl^{\leftrightarrow}_{p,g,\mathbf{0},\mathbf{1}}$

$\mu(Cl^{\leftrightarrow}_{p,\mathbf{0},\mathbf{1}}) = 25\%$
$\mu(NCl^{\leftrightarrow}_{p,\mathbf{0},\mathbf{1}}) = 25\%$



$$\mu(Cl^{\leftrightarrow}_{p,q,\mathbf{0},\mathbf{1}}) = 12.5\%$$
$$\mu(NCl^{\leftrightarrow}_{p,q,\mathbf{0},\mathbf{1}}) = 12.5\%$$


12.5%
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Conjecture

Conjecture 24. *Let* $L = (S, f_1, ..., f_n)$ *be some logic (not necessarily the classical one). If the logic* $L = (S, f_1, ..., f_n)$ *is functionally complete then there exist the asymptotic densities of the class of tautologies as well as the class of non-tautologies.*

Functional completeness and density of truth of intuitionistic logic

$\{\wedge, \neg\}$, $\{\vee, \neg\}$ $\{\rightarrow, \neg\}$ not functionally complete.

We do not have classical definability.

$$(\neg\alpha \vee \neg\beta) \rightarrow \neg(\alpha \wedge \beta)$$

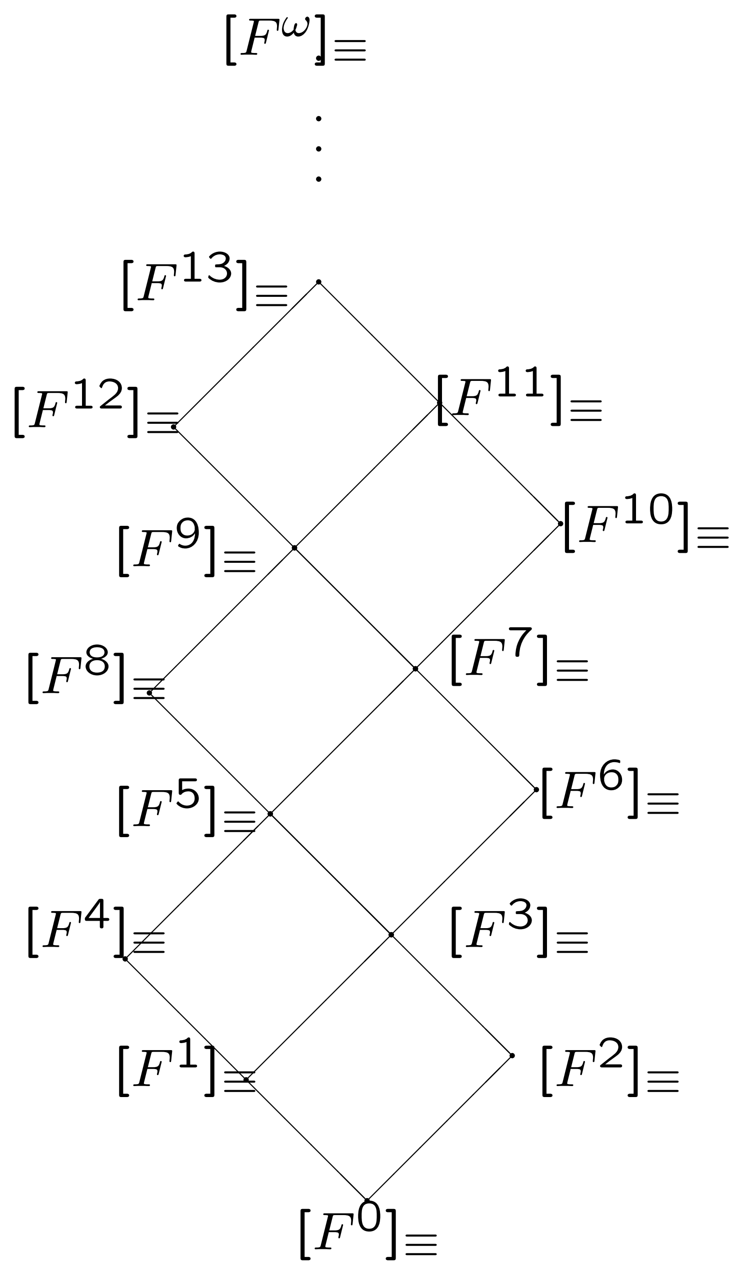
$$(\alpha \wedge \neg\beta) \rightarrow \neg(\alpha \rightarrow \beta)$$
$$(\neg\alpha \vee \beta) \rightarrow (\alpha \rightarrow \beta)$$
$$(\neg\alpha \rightarrow \beta) \rightarrow (\alpha \vee \beta)$$

System $Int_1^{\rightarrow,\vee,\neg}$

$$F^0 = \neg(p \rightarrow p) \tag{17}$$
$$F^1 = p \tag{18}$$
$$F^2 = \neg p \tag{19}$$
$$F^{2n+1} = F^{2n} \vee F^{2n-1} \tag{20}$$
$$F^{2n+2} = F^{2n} \rightarrow F^{2n-1} \tag{21}$$

for $n \geq 1$

Definition 25. $\varphi \equiv \psi$ *if both* $\varphi \rightarrow \psi$ *and* $\psi \rightarrow \varphi$ *are intuitionistic theorems.*



$[F^{\omega}]_{\equiv}$
$[F^{13}]_{\equiv}$
$[F^{12}]_{\equiv}$
$[F^{11}]_{\equiv}$
$[F^{10}]_{\equiv}$
$[F^{9}]_{\equiv}$
$[F^{8}]_{\equiv}$
$[F^{7}]_{\equiv}$
$[F^{6}]_{\equiv}$
$[F^{5}]_{\equiv}$
$[F^{4}]_{\equiv}$
$[F^{3}]_{\equiv}$
$[F^{1}]_{\equiv}$
$[F^{2}]_{\equiv}$
$[F^{0}]_{\equiv}$

System $Int_1^{\rightarrow,\neg}$

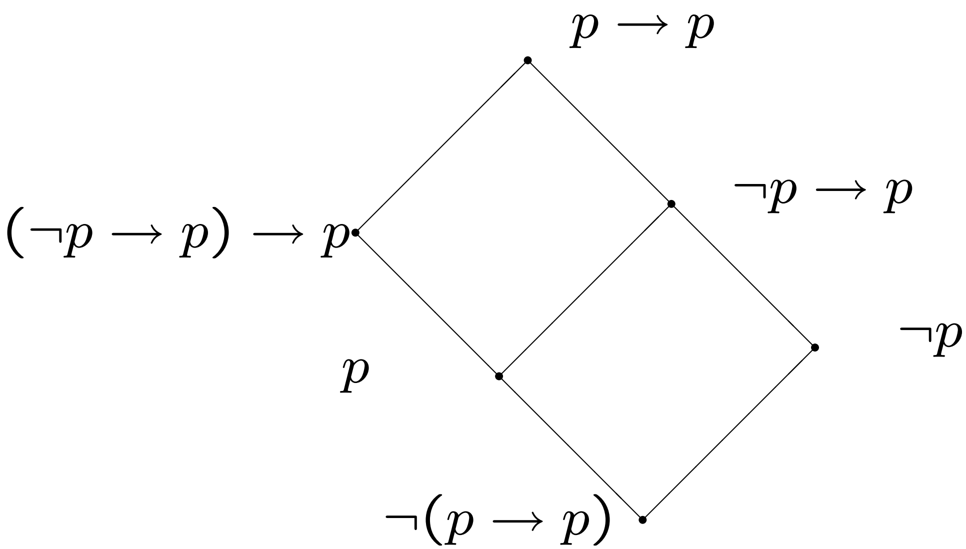
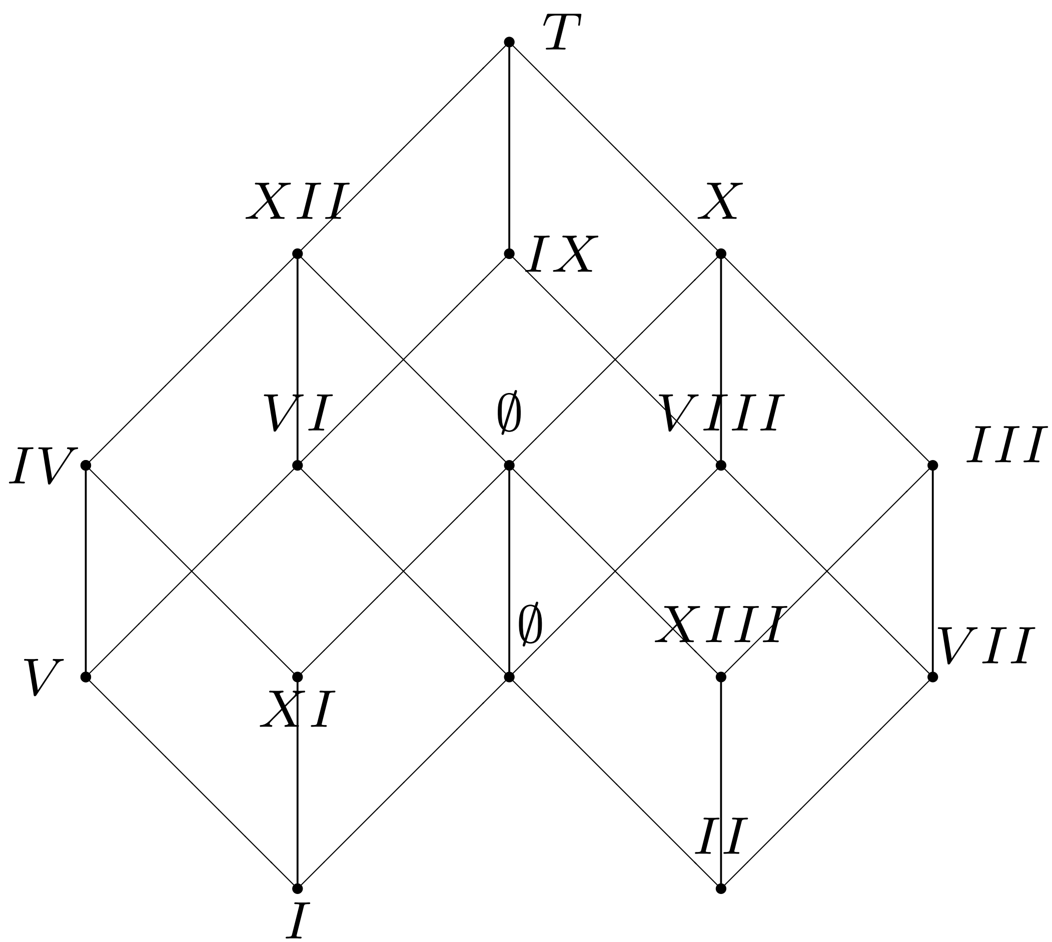


Diagram 7.

System $Int_{2}^{\rightarrow}$



$$\mu(Int_p^{\rightarrow,\neg}) \approx 39.5\%$$
$$\mu(Int_{p,q}^{\rightarrow}) \approx 50.43\%$$

[3] Z. Kostrzycka, M. Zaionc, *Statistics of intuitionistic versus classical logics*, Studia Logica, Vol. 76, Number 3, 2004, pp 307 - 328.

[2] Kostrzycka Z., *On the density of implicational parts of intuitionistic and classical logics*, Journal of Applied Non-Classical Logics, Vol. 13, Number 3, 2003, pp 295-325.

They values for classical logic are as follows:

$$\mu(Cl_p^{\to,\neg}) \approx 42.3\%$$
$$\mu(Cl_{p,q}^{\to}) \approx 51.9\%$$

Theorem 26. *[Relative density] The relative density of intuitionistic tautologies among the classical ones in the language* $Form_1^{\to,\neg}$ *is more than 93 %.*

Theorem 27. *[Relative density] The relative density of intuitionistic tautologies among the classical ones in the language* $Form_2^{\to}$ *is more than 97%.*

Systems $Int_k^{\rightarrow}$ and $Cl_k^{\rightarrow}$

Theorem 28. *Asymptotically (for a large number k of variables), classical tautologies are intuitionistic i.e.*

$$\lim_{k\rightarrow\infty} \frac{\mu^{-}(Int_k^{\rightarrow})}{\mu(Cl_k^{\rightarrow})} = 1$$

where $\mu^{-}(Int_k^{\rightarrow}) = \liminf_{n\rightarrow\infty} \frac{|Int_k^{\rightarrow}|}{|Form_k^{\rightarrow}|}$.

[7] Fournier H., Gardy D., Genitrini A., Zaionc M. *Classical and intuitionistic logic are asymptotically identical*, Lecture Notes in Computer Science 4646, pp. 177-193.

Systems $Int^{\rightarrow}_{p_1,\ldots,p_k,\mathbf{0}}$ and $Cl^{\rightarrow}_{p_1,\ldots,p_k,\mathbf{0}}$

We have a similar result for these logics i.e.

$$\lim_{k\rightarrow\infty} \frac{\mu^{-}(Int^{\rightarrow}_{p_1,\ldots,p_k,\mathbf{0}})}{\mu(Cl^{\rightarrow}_{p_1,\ldots,p_k,\mathbf{0}})} = 1$$

Observation 29. *The logic* $Int^{\rightarrow}_{p_1,\ldots,p_k,\mathbf{0}}$ *is not functionally complete.*

Problem 30. *Is there a density of truth of* $Int^{\rightarrow,\vee}_{p_1,\ldots,p_k,\mathbf{0}}$ *(or* $Int^{\rightarrow,\vee,\neg}_{p_1,\ldots,p_k}$*) logic?*

Problem 31. *Is there a density of truth of* $Int^{\rightarrow,\vee}_{p,\mathbf{0}}$ *(or* $Int^{\rightarrow,\vee,\neg}_{p}$*) logic?*