Normal extensions of some fragment of Grzegorczyk's modal logic

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Abstract

We examine normal extensions of Grzegorczyk's modal logic over the language $\{\rightarrow, \Box\}$ with one propositional variable. Corresponding Kripke frames, including the so-called universal frames, are investigated in the paper. By use of them we characterize the Tarski-Lindenbaum algebras of the logics considered.

1 Grzegorczyk's logic

Syntactically, Grzegorczyk's modal logic **Grz** is obtained by adding to the axioms of classical logic the following modal formulas

$$\begin{array}{ll} (re) & \Box p \rightarrow p \\ (2) & \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \\ (tra) & \Box p \rightarrow \Box\Box p \\ (grz) & \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p \end{array}$$

The logic **Grz** is defined as the set of all consequences of the new axioms by modus ponens, substitution and necessitation (R_G) rules. The last one can be presented by following scheme:

$$(R_G) \quad \frac{\vdash \alpha}{\vdash \Box\alpha}.$$

Semantically, **Grz** logic is characterized by the class of reflexive transitive and antisimmetric Kripke frames which do not contain any infinite ascending chains of distinct points.

Recall, that by a frame we mean a pair $\mathfrak{F} =< W, R >$ consisting of a nonempty set W and a binary relation R on W. The elements of W are called points and xRy is read as 'y is accessible from x'. By $x\uparrow$ we mean the set of successors of x and by $x\downarrow$ -the set of its predecessors.

A model $\mathfrak{M}$ is a triple $< W, R, V >$, where V is a valuation in $\mathfrak{F}$ associating with each variable p a set of $V(p)$ of points in W. $V(p)$ is construed as the set of points at which p is true. By induction on construction of α we define a truth relation '$\models$' in $\mathfrak{F}$. Let $\mathcal{ML}$ be a fixed modal language.

Definition 1.

$$
\begin{aligned}
(\mathfrak{F}, x) \models p \quad & \textit{iff} \quad x \in V(p), \textit{ for every } p \in Var\mathcal{ML} & (1)\\
(\mathfrak{F}, x) \models \alpha \to \beta \quad & \textit{iff} \quad (\mathfrak{F}, x) \models \alpha \textit{ implies } (\mathfrak{F}, x) \models \beta, & (2)\\
(\mathfrak{F}, x) \not\models \bot, & & (3)\\
(\mathfrak{F}, x) \models \Box\alpha \quad & \textit{iff} \quad (\mathfrak{F}, y) \models \alpha \textit{ for all } y \in W \textit{ such that } xRy. & (4)
\end{aligned}
$$

If $\mathfrak{M}$ is known we write $x \models \varphi$ instead of $(\mathfrak{M}, x) \models \varphi$.
φ is valid in a frame $\mathfrak{F}$ if φ is true in all models based on $\mathfrak{F}$.
In this paper we will consider formulas built up from one propositional variable p by means of implication and necessity operator only.

$$
\begin{aligned}
& p \in \mathcal{F}^{\{\to,\Box\}} \\
& \alpha \to \beta \in \mathcal{F}^{\{\to,\Box\}} \text{ iff } \alpha \in \mathcal{F}^{\{\to,\Box\}} \text{ and } \beta \in \mathcal{F}^{\{\to,\Box\}} \\
& \Box\alpha \in \mathcal{F}^{\{\to,\Box\}} \text{ iff } \alpha \in \mathcal{F}^{\{\to,\Box\}}.
\end{aligned}
$$

2 Implication algebras

In this chapter we recall some algebraic notions and facts concerning implication, Boolean and modal algebras (for details see [1]).

Definition 2. *An abstract algebra $\mathcal{A} = (A, \mathbf{1}, \Rightarrow)$ is said to be an* implication algebra provided for all $a, b, c \in A$ the following conditions are satisfied:

$$
\begin{aligned}
a \Rightarrow (b \Rightarrow a) &= \mathbf{1}, & (5)\\
(a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c)) &= \mathbf{1}, & (6)\\
\text{if } a \Rightarrow b = \mathbf{1} \text{ and } b \Rightarrow a = \mathbf{1}, \text{ then } a &= b, & (7)\\
a \Rightarrow \mathbf{1} &= \mathbf{1}, & (8)\\
(a \Rightarrow b) \Rightarrow a &= a. & (9)
\end{aligned}
$$

We shall define a new two-argument operation in any implication algebra $(A, \mathbf{1}, \Rightarrow)$ as follows:

$$a \cup b = (a \Rightarrow b) \Rightarrow b \quad \text{for all } a, b \in A. \tag{10}$$

We also define an order $\leq$ on $(A, \mathbf{1}, \Rightarrow)$ in the usual way:

$$a \leq b \text{ iff } a \Rightarrow \beta = \mathbf{1}. \tag{11}$$

Lemma 3. *In any implication algebra $(A, \mathbf{1}, \Rightarrow)$ and for all $a, b \in A$*

$$a \cup b = l.u.b\{a, b\}, \tag{12}$$

where $\cup$ is defined by (10) and $l.u.b.\{a, b\}$ denotes the least upper bound of $\{a, b\}$ in an ordered set $(A, \leq)$.

Now, we shall define $g.l.b.\{a, b\}$ - the greatest lower bound of $\{a, b\}$. Suppose, there is a zero element $\mathbf{0}$ in an algebra $(A, \mathbf{1}, \Rightarrow)$. So, we can introduce a new one-argument operation of complementation $-$ and a two-argument operation of intersection as follows:

$$
\begin{aligned}
-a &= a \Rightarrow \mathbf{0} \quad \text{for all } a \in A, & (13)\\
a \cap b &= -(-a \cup -b) \quad \text{for all } a, b \in A, & (14)
\end{aligned}
$$

It is obvious that $g.l.b.\{a,b\} = a \cap b$.
We define the following equations:

$$a \Rightarrow -b = b \Rightarrow -a, \tag{15}$$
$$-(a \Rightarrow a) \Rightarrow b = \mathbf{1}. \tag{16}$$

The connection between implication algebras and Boolean algebras is established by the following lemma (see [1]):

Lemma 4. *If $(A, \mathbf{0}, \mathbf{1}, \Rightarrow, -)$ is an abstract algebra such that $(A, \mathbf{1}, \Rightarrow)$ is an implication algebra with zero element and the equations (15), (16) hold, then $(A, \mathbf{0}, \mathbf{1}, \Rightarrow, \cup, \cap, -)$, where the operations $\cup, \cap$ are defined by (10), (14), is a Boolean algebra.*

Definition 5. *By a modal algebra we mean an algebra $\mathcal{A} = (A, \mathbf{0}, \mathbf{1}, \Rightarrow, \cup, \cap, -, l)$, where $(A, \mathbf{0}, \mathbf{1}, \Rightarrow, \cup, \cap, -)$ is a Boolean algebra and l is a unary operation satisfying the conditions:*

$$l\mathbf{1} = \mathbf{1}, \tag{17}$$
$$l(a \cap b) = la \cap lb. \tag{18}$$

3 Normal extensions of Grz

This section will be concerned with normal extensions of **Grz** determined by appropriate Kripke frames with finite depth.

Definition 6. *A frame $\mathfrak{F}$ is of depth $n < \omega$ if there is a chain of n points in $\mathfrak{F}$ and no chain of more than n points exists in $\mathfrak{F}$.*

For $n > 0$, let J_n be an axiom that says any strictly ascending partial-ordered sequence of points is of length n at most, i.e.,that there exist no points $x_1, x_2, ..., x_n$ such that x_{n+1} is accessible from x_i for $i = 1, 2, ..., n$. The formulas J_n are well known (see for example [2] p.42) and are defined inductively as follows[1]

Definition 7.

$$J_1 = \Diamond\Box p_1 \to p_1,$$
$$J_{n+1} = \Diamond(\Box p_{n+1} \wedge \sim J_n) \to p_{n+1}.$$

We will consider the logics $\mathbf{Grz}^{\leq n} = \mathbf{Grz} \oplus J_n$. They contain the logic **Grz** and the following inclusions hold:

$$\mathbf{Grz} \subset ... \subset \mathbf{Grz}^{\leq n} \subset \mathbf{Grz}^{\leq n-1} \subset ... \subset \mathbf{Grz}^{\leq 2} \subset \mathbf{Grz}^{\leq 1}. \tag{19}$$

To characterize the logics $\mathbf{Grz}^{\leq n}$, we describe the appropriate Tarski-Lindenbaum algebras $\mathbf{Grz}^{\leq n}/_{\equiv}$.

Definition 8. *$\alpha \equiv \beta$ iff $\alpha \to \beta \in \mathbf{Grz}^{\leq n}$ and $\beta \to \alpha \in \mathbf{Grz}^{\leq n}$ for $n = 1, 2, ..., n$.*

[1]The formulas J_n are defined in the full language. In the language $\mathcal{F}^{\{\to,\Box\}}$ we can find the analogous formulas. We will see in Section 5 the formula A_{2n+1} plays the role of formula J_n (see Lemma 29).

This equivalence relation depends on n. In fact we have n different equivalence relations; one for each logic $\mathbf{Grz}^{\leq n}$.

Definition 9. $\mathbf{Grz}^{\leq n}/_{\equiv} = \{[\alpha]_{\equiv},\, \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}\}$

Definition 10. *The order of classes* $[\alpha]_{\equiv}$ *is defined as*
$[\alpha]_{\equiv} \leq [\beta]_{\equiv}$ *iff* $\alpha \rightarrow \beta \in \mathbf{Grz}^{\leq n}$ *for* $n = 1, 2, ..., n$.

Lemma 11. *For any algebra* $\mathbf{Grz}^{\leq n}/_{\equiv}$ *the following orders hold:*

$$[\Box p]_{\equiv} \leq [\alpha]_{\equiv} \text{ for any } \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}, \tag{20}$$
$$[\alpha]_{\equiv} \leq [p \rightarrow p]_{\equiv} \text{ for any } \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}, \tag{21}$$

where $\leq$ *is defined in Definition 10.*

Proof. Obvious.
We see that the class $[\Box p]_{\equiv}$ behaves as $\mathbf{0}$ of the lattice $\mathbf{Grz}^{\leq n}/_{\equiv}$, while $[p \rightarrow p]_{\equiv}$ as $\mathbf{1}$.

Lemma 12. *Every algebra* $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \rightarrow)$ *is an implication algebra including* $\mathbf{0} = [\Box p]_{\equiv}$.

Proof. Since the implication $\rightarrow$ is just classical one, the conditions (5,6,7,8,9) are fulfilled. □
After introducing the new operations $\vee, \sim, \wedge$ defined analogously to (10,13,14) we have:

Lemma 13. *Every algebra* $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \rightarrow, \vee, \wedge, \sim)$ *is a Boolean algebra.*

Proof. It follows from Lemma 12 and 4. □

Lemma 14. *Every algebra* $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \rightarrow, \vee, \wedge, \sim, \Box)$ *is a modal algebra.*

Proof. It follows from Lemma 13 and from the fact the $\Box$ fulfills the conditions (17) and (18). □

4 Universal models

In this section we review some of the standard facts on canonical, filtrated and universal models (for details see [2]). First, let us to recall the notion of canonical frame. Roughly speaking it is a frame built over a language. Points x_i in canonical frame are maximal consistent sets of formulas (for details see [2]). Hence $x_i = (\Gamma_i, \Delta_i)$ and $\phi \in \Gamma_i$ iff $x_i \models \phi$ and $\varphi \in \Delta_i$ iff $x_i \not\models \varphi$.

Definition 15. *Let* $\mathfrak{F}_L =< W_L, R_L >$ *be a frame such that* W_L *is the set of all maximal L-consistent tableaux and for any* $x_1 = (\Gamma_1, \Delta_1)$ *and* $x_2 = (\Gamma_2, \Delta_2)$ *in* W_L*:* $x_1 R_L x_2$ *iff* $\{\phi : \Box\phi \in \Gamma_1\} \subseteq \Gamma_2$.

Define the valuation V_L in $\mathfrak{F}_L$ for the variable p as follows:

$$V_L(p) = \{(\Gamma, \Delta) \in W_L : p \in \Gamma\}.$$

The resulting model $\mathfrak{M}_L =< \mathfrak{F}_L, V_L >$ is called the canonical model for L.

Grzegorczyk's logic is not canonical. Canonical frame $\mathfrak{F}_{Grz}$ is reflexive and transitive, but can contain proper clusters. To avoid it the selective filtration is used.

Let $\sum$ be a set of formulas closed under their subformulas.

Definition 16.

$$x \sim_\Sigma y \quad \textit{iff} \quad ((\mathfrak{M}, x) \models \phi \quad \textit{iff} \quad (\mathfrak{M}, y) \models \phi), \quad \textit{for every } \phi \in \Sigma$$

Definition 17. *A filtration of* $\mathfrak{M} =< W, R, V >$ *through* Σ *is a model* $\mathfrak{N} =< Z, S, U >$ *such that: (i)* $Z = \{[x] : x \in W\}$*,*
(ii) $U(p) = \{[x] : x \in V(p)\}$ *for every* $p \in \Sigma$*,*
(iii) xRy *implies* $[x]S[y]$ *for all* $x, y \in W$*,*
(iv) if $[x]S[y]$ *then* $y \models \phi$ *whenever* $x \models \Box\phi$ *for* $x, y \in W$ *and* $\Box\phi \in \Sigma$

Let $\mathfrak{M}_{Grz}$ be the canonical and filtrated model for **Grz**. The following lemma is proved in [2]:

Lemma 18. *Suppose* $\Box\phi \in \sum$*,* $x \models \phi$ *and* $x \not\models \Box\phi$ *for some point* x *in* $\mathfrak{M}_{Grz}$*. Then there is a point* $y \in x\uparrow$ *such that* $y \not\models \phi$ *and* $z \sim_\Sigma x$ *for no* $z \in y\uparrow$*.*

From the above lemma it follows that the filtrated canonical model for **Grz** is a finite partial order without proper clusters.

Definition 19. *A modal general frame is a triple* $\mathfrak{F} =< W, R, P >$ *in which* $< W, R >$ *is an ordinary Kripke frame and* P*, a set possible values in* $\mathfrak{F}$*, is a subset of* 2^W *containing* $\emptyset$ *and closed under* $\cap, \cup$ *and operations* $\supset, \Box$ *as follow:*

$$X \supset Y = (W - X) \cup Y,$$
$$\Box X = \{x \in W : \forall y \in W(xRy \Rightarrow y \in X)\}$$

The algebra $< P, \cap, \cup, \rightarrow, \emptyset, \Box >$ is a modal algebra and is called the dual algebra of $\mathfrak{F}$ and denoted by $\mathfrak{F}^+$. A valuation V is defined in the same way as for Kripke models and $V(\phi) = \{x \in W : x \models \phi\}$.

Definition 20. *The general frame associated with the canonical model* $\mathfrak{M}_L$ *is called universal frame and denoted by* $\gamma\mathfrak{F}_L =< W_L, R_L, P_L >$*.*

The connection between Tarski - Lindenbaum's algebras and dual algebras is showed in the following theorem:

Theorem 21. *For every normal modal logic* L *the Tarski-Lindenbaum algebra* $L/_\equiv$ *is isomorphic to the dual* $\gamma\mathfrak{F}_L^+$ *of the universal frame* $\gamma\mathfrak{F}_L$*. The isomorphism is a map* f *defined by* $f([\phi]_\equiv) = V_L(\phi)$*.*

5 Building the universal frame for $\mathbf{Grz}^{\leq n}$

Now, we can approach the main problem. We will build the universal frame $\gamma\mathfrak{F}_{Grz}^{\leq n}$ generated by one variable and show that for any $n \in \mathbb{N}$ the algebra $\mathbf{Grz}^{\leq n}/_\equiv$ is finite. The length of formula is defined in a normal way:

Definition 22.

$$\begin{aligned} l(p) &= 1 \\ l(\Box\phi) &= 1 + l(\phi) \\ l(\phi \to \psi) &= l(\phi) + l(\psi) + 1 \end{aligned}$$

Definition 23. *A point x in a frame $\mathfrak{F}$ is of depth d iff the subframe generated by x is of depth d.*

Lemma 24. *Let $\gamma\mathfrak{F}_1 =< W^{\leq n}_{Grz} \cup \{x'_n\}, R^{\leq n}_{Grz}, P^{\leq n}_{Grz} >$ and $\gamma\mathfrak{F}_2 =< W^{\leq n}_{Grz}, R^{\leq n}_{Grz}, P^{\leq n}_{Grz} >$ be two universal frames for $Grz^{\leq n}$, where x'_n is the point of depth 1 such that $x'_n \models p$. Suppose the valuations of p do not differ in $\gamma\mathfrak{F}_1$ and $\gamma\mathfrak{F}_2$ at the same points. For any $\alpha \in \mathcal{F}^{\{\to,\Box\}}$, for any $x_i \in W^{\leq n}_{Grz}$ the following equivalence holds:*

$$(\gamma\mathfrak{F}_1, x_i) \models \alpha \text{ iff } (\gamma\mathfrak{F}_2, x_i) \models \alpha. \tag{22}$$

Proof. Let $(x_1, x_2, ..., x_n)$ be any chain of points in $W^{\leq n}_{Grz}$. The proof is by induction on the depth i for $i = 1, ..., n$ of points x_{n-i+1}. For $i = 1$ it is obvious the point x'_n is not accessible to any other point of depth 1 and then (22) holds trivially.
Suppose (22) holds at points of depth i. Now we use induction on the length of α. If $\alpha = p$ then (22) is obvious. Suppose (22) is true for α such that $l(\alpha) \leq k$ at the point x_{n-i-1}. We show (22) holds for α of length $k + 1$ at the same point. We consider two cases:

1. Let $\alpha = \alpha_1 \to \alpha_2$ and $(\gamma\mathfrak{F}_1, x_{n-i-1}) \not\models \alpha$. Then $(\gamma\mathfrak{F}_1, x_{n-i-1}) \models \alpha_1$ and $(\gamma\mathfrak{F}_1, x_{n-i-1}) \not\models \alpha_2$. From inductive hypothesis we have $(\gamma\mathfrak{F}_2, x_{n-i-1}) \models \alpha_1$ and $(\gamma\mathfrak{F}_2, x_{n-i-1}) \not\models \alpha_2$ and hence $(\gamma\mathfrak{F}_2, x_{n-i-1}) \not\models \alpha$. The proof of reverse implication is analogous.
2. Let $\alpha = \Box\alpha_1$ and $(\gamma\mathfrak{F}_1, x_{n-i-1}) \not\models \Box\alpha_1$.
   (a) Suppose it is because $(\gamma\mathfrak{F}_1, x_{n-i-1}) \not\models \alpha_1$. From inductive hypothesis we have $(\gamma\mathfrak{F}_2, x_{n-i-1}) \not\models \alpha_1$. Then $(\gamma\mathfrak{F}_2, x_{n-i-1}) \not\models \Box\alpha_1$.
   (b) Suppose we have $(\gamma\mathfrak{F}_1, x_{n-i-1}) \models \alpha_1$ and for some $l \leq i$ holds $(\gamma\mathfrak{F}_1, x_{n-l}) \not\models \alpha_1$. The point x_{n-l} must differ from x'_n because at x'_n every formula $\alpha \in \mathcal{F}^{\{\to,\Box\}}$ is true (it is the last point in the frame $\gamma\mathfrak{F}_1$). From inductive hypothesis we have $(\gamma\mathfrak{F}_2, x_{n-l}) \not\models \alpha_1$. Then $(\gamma\mathfrak{F}_2, x_{n-i-1}) \not\models \Box\alpha_1$.

   If $(\gamma\mathfrak{F}_1, x_{n-i-1}) \models \Box\alpha_1$ the proof is obvious.

□

From Lemma 11 we deduce that if the last point validates p (and $\Box p$), then it validates all formulas from $\mathcal{F}^{\{\to,\Box\}}$. On the base of Lemma 24 we need only consider universal frames with the last points not validating p. It coincides with consistency of universal frames (see (3) in Definition 1). Consistency, in general involves $Grz^{\leq n} \neq \mathcal{F}^{\{\to,\Box\}}$.

Corollary 25. *The universal frame $\gamma\mathfrak{F}^{\leq 1}_{Grz}$ consists of one point x such that $x \not\models p$.*

Proof. Every two points x and x' not validating p are equivalent to each other and after using the selective filtration we obtain one-element frame. □

Lemma 26. *The universal frame $\gamma\mathfrak{F}^{\leq 2}_{Grz}$ consists of two points x_1 and x_2 such that $x_1 R x_2$, $x_2 \not\models p$ and $x_1 \models p$.*

Proof. Because of Corollary 25 it is enough to show that does not exist a point x_1' such that $x_1'Rx_2$ and $x_1' \not\models p$. We show that if such a point exists it will be equivalent to the point x_2. We prove by induction on the length of α that for all k and $\alpha \in \mathcal{F}^{\{\to,\Box\}}$

$$x_1' \models \alpha \text{ iff } x_2 \models \alpha \tag{23}$$

For $k = 1$ it is obvious that (23) is fulfilled. Assume (23) holds for k; we will prove it for $k + 1$.

1. Let $\alpha = \alpha_1 \to \alpha_2$ and $x_2 \not\models \alpha$. That means $x_2 \models \alpha_1$ and $x_2 \not\models \alpha_2$. From assumption we have $x_1' \models \alpha_1$ and $x_1' \not\models \alpha_2$ which gives us $x_1' \not\models \alpha$.
2. Let $\alpha = \Box\alpha_1$ and $x_1' \models \Box\alpha_1$. $x_1'Rx_2$ and hence $x_2 \models \Box\alpha_1$. Suppose $x_1' \not\models \Box\alpha_1$. If $x_1' \not\models \alpha_1$, from inductive assumption we have $x_2 \not\models \alpha_1$ and so $x_2 \not\models \Box\alpha_1$. If $x_1' \models \alpha_1$ but $x_2 \not\models \alpha_1$ then we have a contradiction with the inductive assumption.

After using the selective filtration with respect to the set $\mathcal{F}^{\{\to,\Box\}}$ we identify the points x_1' and x_2. □

Below in Diagram 1 we present both the frame $\gamma\mathfrak{F}^{\leq 2}_{Grz}$ and the Tarski-Lindenbaum algebra $\mathbf{Grz}^{\leq 2}$ being isomorphic to the dual algebra $\mathfrak{F}^{\leq 2+}_{Grz}$.

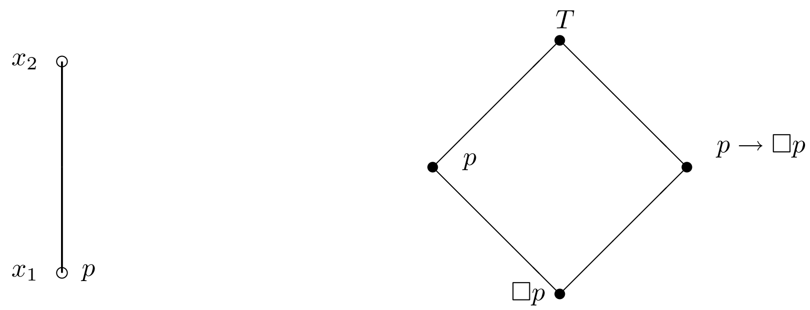

Diagram 1

Lemma 27. *The universal frame $\gamma\mathfrak{F}^{\leq 3}_{Grz}$ consists of three-element chain (x_1, x_2, x_3) such that $x_2 \not\models p$, $x_1 \models p$ and $x_3 \models p$.*

Proof. Analogous to the proof of Lemma 26. □

The diagrams of $\gamma\mathfrak{F}^{\leq 3}_{Grz}$ and the Tarski-Lindenbaum algebra $\mathbf{Grz}^{\leq 3}$ are the following:

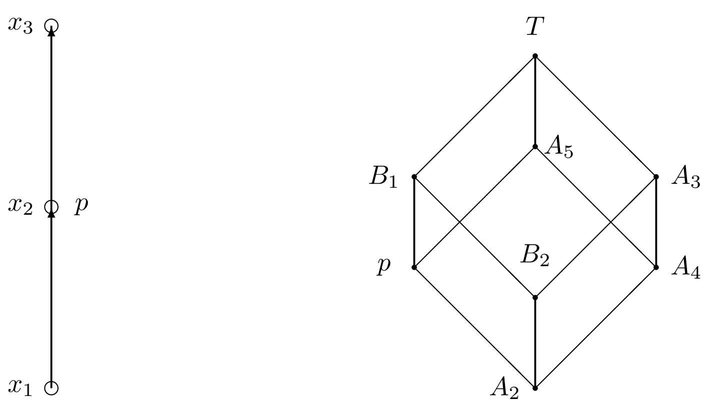

Diagram 2.

where

$$\begin{array}{rcl} A_1 & = & [p]_{\equiv} \\ A_2 & = & \Box A_1 \\ A_3 & = & A_1 \to A_2 \\ A_4 & = & \Box A_3 \\ A_5 & = & A_3 \to A_4 \\ B_1 & = & A_4 \to A_2 \\ B_2 & = & A_5 \to A_2 \end{array}$$

The same reasoning can be applied in the case of building the universal frame with depth n.

Lemma 28. *The universal frame $\mathfrak{F}^{\leq n}_{Grz}$ is an n -element chain $(x_1, x_2, ..., x_n)$ such that for any $k < n/2$:*

$$x_{n-2k} \not\models p \ \ for \ \ k \geq 0, \tag{24}$$

$$x_{n-(2k-1)} \models p \ \ for \ \ k \geq 1. \tag{25}$$

Definition 29.

$$A_1 = p, \quad A_{2n} = \Box A_{2n-1}, \quad A_{2n+1} = A_{2n-1} \to A_{2n}, \quad for \quad n \geq 1.$$

Lemma 30. *Let $\gamma\mathfrak{F}^{\leq n}_{Grz}$ be the universal frame for $\mathbf{Grz}^{\leq n}$. For any $k = 0, ..., n-1$:*

$$x_{n-k} \uparrow \ \ \models \ \ A_{k'} \ \ for \ any \ \ k' \geq 2k+3. \tag{26}$$

Proof. By induction on k. If $k = 0$ then the point x_n is the last point in the chain $(x_1, ..., x_n)$. From Lemma 28, $x_n \not\models p$ and hence $x_n \not\models \Box p$. This gives us $x_n \models A_3$. It is easy to notice that $x_n \models A_{k'}$ for $k' \geq 3$.
Assuming (26) to hold for points of depth $\leq k$, we have $x_{n-k} \uparrow\models A_{k'}$ for $k' \geq 2k+3$ and also $x_{n-k} \uparrow\models A_{2k+3}$. We will prove $x_{n-k-1} \models A_{2k+5}$. If not, then $x_{n-k-1} \models A_{2k+3}$ and $x_{n-k-1} \not\models \Box A_{2k+3}$. Hence there is a point $x' \in x_{n-k-1} \uparrow$ such that $x' \not\models A_{2k+3}$, but it is a contradiction. From inductive hypothesis we have also $x_{n-k-1} \uparrow\models A_{k'}$ for $k' \geq 2k+5$. $\Box$

Lemma 31. *Let $\gamma\mathfrak{F}^{\leq n}_{Grz}$ be the universal frame for $\mathbf{Grz}^{\leq n}$. Then*

$$x_{n-2k} \models A_{4k'+3} \ \ and \ \ x_{n-2k} \not\models A_{4k'+1} \quad for \ any \ \ 0 \leq k' \leq k \ \ and \ \ 1 \leq n-2k \leq n, \tag{27}$$

$$x_{n-(2k-1)} \models A_{4k'+1} \ \ and \ \ x_{n-(2k-1)} \not\models A_{4k'+3} \quad for \ any \ \ 0 \leq k' \leq k \ \ and \ \ 1 \leq n-(2k-1) \leq n, \tag{28}$$

Proof. We use double induction with respect to the k and k'. Let $k = 0$. Then $k' = 0$ and $x_n \not\models p$ and $x_n \models A_3$. We obtained (27). If $k = 1$ then $x_{n-1} \models p$, $x_{n-1} \not\models \Box p$ and hence $x_{n-1} \not\models A_3$. We obtained (28). Assume (27) and (28) hold for some k. We show they hold for $k+1$. Assume now they hold for some $k' \leq k$ and take $k'+1$ such that $k'+1 \leq k$. Let us consider the formula $A_{4k'+7} = A_{4k'+5} \to \Box A_{4k'+5}$. We will prove $x_{n-(2k+2)} \not\models A_{4k'+5}$. We know that $x_{n-(2k+2)} \models A_{4k'+3}$ and $x_{n-(2k+2)} \not\models \Box A_{4k'+3}$ because $x_{n-(2k-1)} \not\models A_{4k'+3}$. So, $x_{n-(2k+2)} \models A_{4k'+7}$ and also $x_{n-(2k+2)} \not\models A_{4k'+5}$. The proof of (28) proceeds similarly. $\Box$

Corollary 32. *Let* $\gamma\mathfrak{F}^{\leq n}_{Grz}$ *be the universal frame for* $\mathbf{Grz}^{\leq n}$*. For any* $k = 0, 1, ..., n-1$*:*

$$\max\{k' : x_{n-k} \not\models A_{2k'+1}\} = k. \tag{29}$$

Corollary 33. *Let* $\gamma\mathfrak{F}^{\leq n}_{Grz}$ *be the universal frame for* $\mathbf{Grz}^{\leq n}$*. For any* $k = 0, 1, ..., n-1$*:*

$$x_{n-k} \not\models A_{2k'+5} \to A_{2k'+1} \text{ iff } k' = k. \tag{30}$$

Because considered frames are 1- generated they are also atomic (see [2], p.270) that are frames in which every point is an atom. The class $[\phi]$ is an atom in a universal frame if there is only one point $x = (\Gamma, \Delta)$ such that $\phi \in \Gamma$. In others words the formula ϕ is possible only at one point.

Theorem 34. *The following classes are atoms in every universal frame* $\gamma\mathfrak{F}^{\leq n}_{Grz}$*:*

$$(A_{2k+5} \to A_{2k+1}) \to A_2 \quad for \quad k = 0, 1, ..., n-1.$$

Proof. In the universal frame $\gamma\mathfrak{F}^{\leq n}_{Grz}$ for any $k \leq n$ we have: $x_k \not\models A_2$. So, from Corollary 33 we have the point x_{n-k} is the only point at which the formula $(A_{2k+5} \to A_{2n+1}) \to A_2$ is true. □

Corollary 35. *Every algebra* $\mathbf{Grz}^{\leq n}/_{\equiv}$ *consists of* 2^n *equivalence classes generated by* n *atoms.*

In the picture below the universal frame $\gamma\mathfrak{F}^{\leq n}_{Grz}$ with listed atoms is presented.

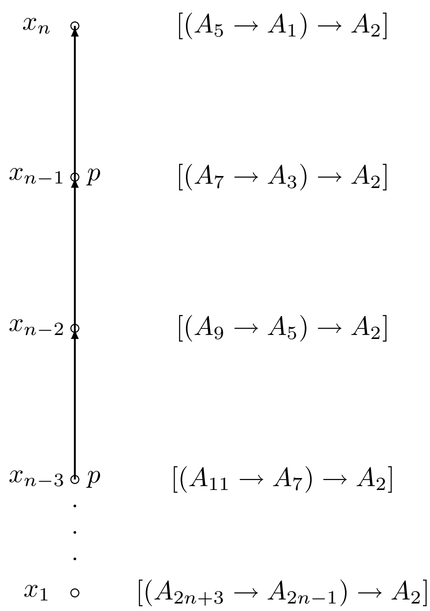

Diagram 3.

Diagram 4 presents the rule of raising of the quotient algebra $\mathbf{Grz}^{\leq n}/_{\equiv}$. More exactly - the whole algebra $\mathbf{Grz}^{\leq 4}/_{\equiv}$ is drawn with the one cube being a part of $\mathbf{Grz}^{\leq 5}/_{\equiv}$. The diagram of $\mathbf{Grz}^{\leq 5}/_{\equiv}$ consists of four analogous cubes not being marked in the picture. The classes of atoms are however listed.

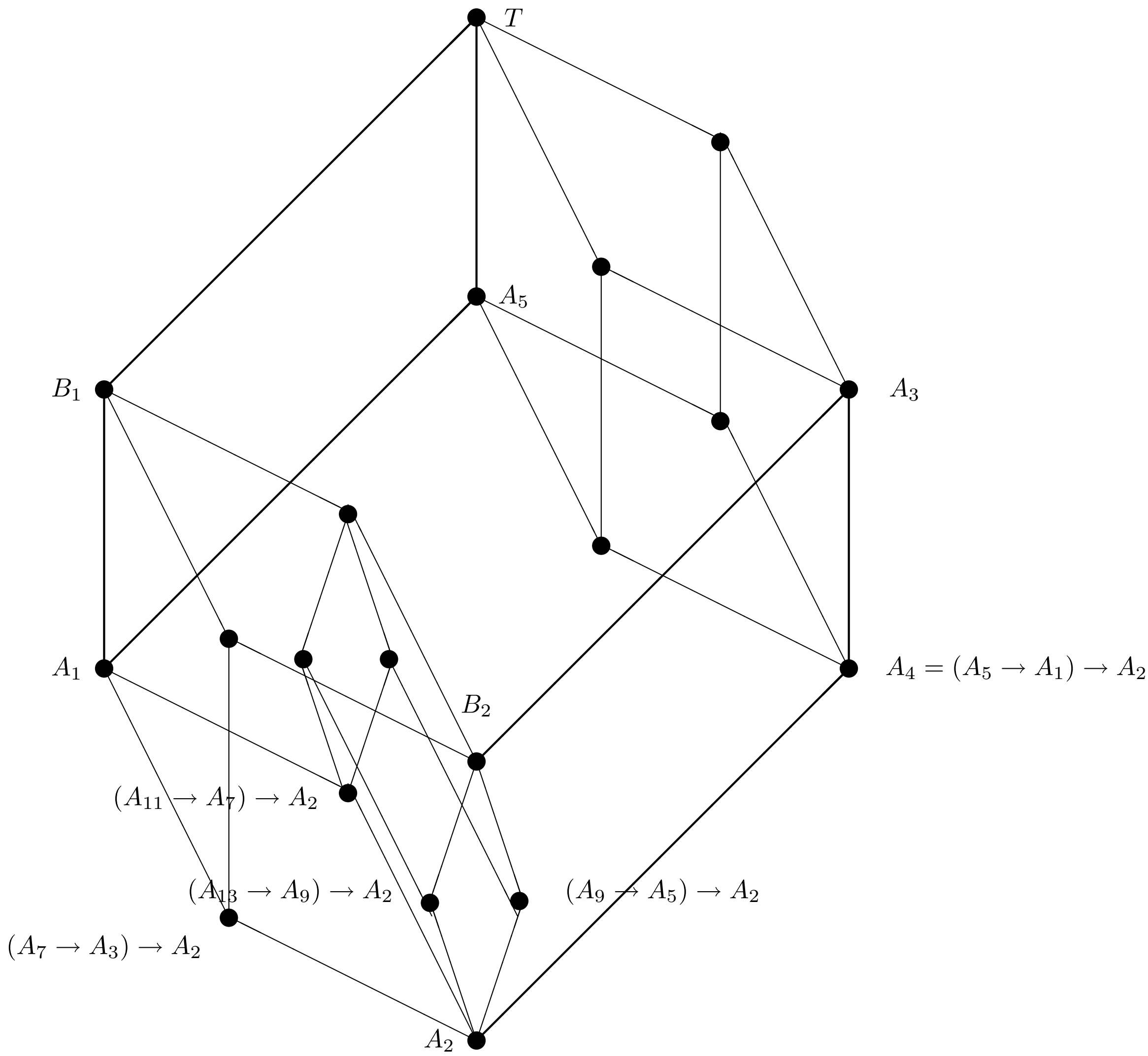

Diagram 4.

Let

$$(sc) \quad \Box(\Box p \to q) \vee \Box(\Box q \to p).$$

It is well known linear Grzegorczyk's logic **Grz.3** = **Grz** $\oplus$ sc is characterized by the linear frame $< \omega, \leq >$.

Observation 36. *The* $\{\to, \Box\}$ *fragment of Grzegorczyk's logic over one variable is the same as the appropriate fragment of linear Grzegorczyk's logic.*

References

[1] Rasiowa, H. (1974) An Algebraic Approach to Non-classical Logics, PWN, Warszawa.

[2] Chagrow A., Zakharyaschev M., (1997) Modal Logic, Oxford Logic Guides 35.