On formulas of one variable in $NEXT(KTB)$

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Abstract

In this paper we consider formulas written in one variable in the normal logic $\mathbf{T_2} = \mathbf{KTB} \oplus \Box^2 p \to \Box^3 p$. We present some special model for $\mathbf{T_2}$ to construct infinitely many non-equivalent formulas written in one variable and some family of models to construct continuum of logics over $\mathbf{T_2}$ * .

1 Introduction

The Brouwer system known as **KTB** logic is an example of non-transitive logic, because is characterized by the class of reflexive symmetric and non-transitive frames. The non-transitivity of frames involves there are very few results concerning this logic and its extension. One of family of extension of **KTB** logic is the family of logics $\mathbf{T_n} = \mathbf{KTB} \oplus (4_n)$, where

$$(4_n) \quad \Box^n p \to \Box^{n+1} p$$

is the axiom of n-transitivity.
We see that $\mathbf{T_1} = \mathbf{S5}$ and the following inclusions hold:

$$\mathbf{KTB} \subset ... \subset \mathbf{T_{n+1}} \subset \mathbf{T_n} \subset ... \subset \underline{\mathbf{T_2}} \subset \mathbf{T_1} = \mathbf{S5}. \tag{1}$$

As it is known each logic $\mathbf{T_n}$ is canonical, because adding the axiom (4_n) involves the following first order condition on frames:

$$(tran_n) \quad \forall_{x,y}(\text{if } \ xR^{n+1}y \ \text{ then } \ xR^n y)$$

where the relation of n-step accessibility is defined inductively as follows:

$$xR^0 y \quad \text{iff} \quad x = y$$
$$xR^{n+1} y \quad \text{iff} \quad \exists_z (xR^n z \ \wedge \ zRy)$$

2 The reduction of models for $\mathbf{T_2}$ logic

Let us consider 1-generated fragment of $\mathbf{T_2}$ logic. That means the considered language consists of formulas built up from one variable. The set of such modal formulas is denoted by $\mathcal{ML}_1$. We will show, the the restriction to the set $\mathcal{ML}_1$ involves significant simplification of $\mathbf{T_2}$ models.
To make the reduction of models we introduce some congruence relation.
Let $\mathfrak{M} = \langle W, R, V \rangle$ be a Kripke $\mathbf{T_2}$ model and $\sim$ an equivalence relation on W defined as follows:

$$\forall_{x,y \in W}\ x \sim y \quad \text{iff} \quad \forall_{\alpha \in \mathcal{ML}_1}(x \models \alpha \iff y \models \alpha).$$

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Definition 1. *The new quotient model is a model* $[\mathfrak{M}] = \langle [W], [R], [V] \rangle$ *such that:*
(i) $[W] = \{[x] : x \in W\}$*,*
(ii) $[R] = \{\langle [x], [y] \rangle : \exists_{x_1 \in [x]} \exists_{y_1 \in [y]} x_1 R y_1\}$
(iii) $[V](p) = [V(p)]$ *for every variable* p*.*

Before we start the reduction we have to establish the cohesion of every frame. It is shown in [4] that every Kripke frame $\mathfrak{F} = \langle W, R \rangle$ might be decomposed into frames such that:

$$\forall_{x,y \in W} \ \exists_n \ xR^n y$$

In this paper we consider only cohesive frames. Let us observe that if we add the condition ($tran_2$) then we immediately have:

Observation 2. *Let* $\mathfrak{M} = \langle W, R, V \rangle$ *be a KTB model. Then the condition* ($tran_2$) *involves that*

$$\forall_{x,y \in W} \ (xRy \vee xR^2 y)$$

Now, we are ready to start the reduction.

Lemma 3. *Let* $\mathfrak{M} = \langle W, R, V \rangle$ *be a KTB model such that*

$$\begin{array}{l} (i) \ \forall_{x \models p} \ \ \exists_{y \not\models p} \ xRy \\ and \\ (ii) \ \forall_{y \not\models p} \ \ \exists_{x \models p} \ xRy \end{array}$$

After reduction we obtain the following quotient model: $[\mathfrak{M}] = \langle [W], [R], [V] \rangle$ *such that:* $[W] = \{[x], [y]\}$, $\quad [R] = \{\langle [x], [x] \rangle, \langle [y], [y] \rangle, \langle [x], [y] \rangle, \langle [y], [x] \rangle\}, [x] \models p, \quad [y] \not\models p$.

Proof. We show that every point $x \in W$ such that (i) validates the same formulas as well as every point $y \in W$ such that (ii). Formally, let us define two sets:

$$X := \{x \in W : \ x \models p \text{ and } \exists_{x' \in W}(xRx' \text{ and } x' \not\models p)\} \tag{2}$$
$$Y := \{y \in W : \ y \not\models p \text{ and } \exists_{y' \in W}(yRy' \text{ and } y' \models p)\} \tag{3}$$

We show that

$$\forall_{x,y \in X} \ x \sim y \quad \text{and} \quad \forall_{z,u \in Y} \ z \sim u \tag{4}$$

We use induction. The length $|\alpha|$ of formula α is defined in the conventional way. We see that for $\alpha = p$ this holds. Suppose (4) holds for formulas of the length less than or equal to n. The cases of implication, negation, conjunction and disjunction are the trivial ones. Let $\alpha = \Box\beta$. Let $x, y \in X$. Suppose $x \not\models \Box\beta$. Then there exists $x' \in W$ such that xRx' and $x' \not\models \beta$. If $x' \in X$, then by the induction hypothesis $y \not\models \beta$ and also $y \not\models \Box\beta$. If $x' \notin X$ then $x' \in Y$. From induction hypothesis there is some $y' \in Y$ such that $y' \not\models \beta$ and $y'Ry$. But then $y \not\models \Box\beta$.

□

Lemma 4. *Let* $\mathfrak{M} = \langle W, R, V \rangle$ *be a* T_2 *model such that*

$$\exists_{x \in W} \ x \models \Box^2 p \tag{5}$$

Then reduction gives us the following model: $[\mathfrak{M}] = \langle [W], [R], [V] \rangle$ *such that:* $[W] = \{[x]\}$, $\quad [R] = \{\langle [x], [x] \rangle\}$, $[x] \models p$.

Proof. From the assumption (5) and Observation 2 we conclude that

$$\forall_{y\in W}\ y \models p$$

Then of course

$$\forall_{y\in W}\ y \models \Box^2 p$$

Of course such model after reduction gives us the 1-element quotient model $[\mathfrak{M}]$. $\Box$

Corollary 5. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a T_2 model such that*

$$\exists_{x\in W}\ x \models \Box^2\neg p \tag{6}$$

By reduction we get the following model: $[\mathfrak{M}] = \langle [W], [R], [V]\rangle$ such that: $[W] = \{[x]\}$, $[R] = \{\langle [x], [x]\rangle\}$, $[x] \not\models p$.

Proof. Analogous to the proof of Lemma 4.

Because of Lemma 4 and Corollary 5 it seems to be reasonable to consider as a non-trivial T_2 models only these ones in which

$$\forall_{x\in W}\ x \not\models \Box^2 p \quad \text{and} \quad \forall_{x\in W}\ x \not\models \Box^2\neg p \tag{7}$$

Lemma 6. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a T_2 model such that $|W| > 2$, (7) holds and there exist at least two points x_1, x_2 such that $x_1 R^n x_2$ for some $n \geq 1$. Then the following two conditions do not hold together:*

$$x_1 \models \Box p \tag{8}$$

$$x_2 \models \Box\neg p \tag{9}$$

Proof. Obvious.

From the above consideration we conclude that the non-trivial T_2 models are these ones in which there is at least one point validating $\Box p$ (or $\Box\neg p$).

Let us define the following model (see Diagram 1):

Definition 7. $\mathfrak{M} = \langle W, R, V\rangle$, *where*

$$W := \{x_1, x_2\} \cup \{y_i, i \geq 1\}$$

The relation R is reflexive, symmetric, 2-step accessible and:

$$x_1 R x_2, \quad x_2 R y_i \ \textit{ for any } \ i \geq 1$$
$$y_i R y_j \ \textit{ if } \ |i-j| \leq 1 \ \textit{ for any } \ i, j \geq 1$$

The valuation is the following:

$$V(p) := \{x_1, x_2\} \cup \{y_{2m+1}, m \geq 1\}.$$

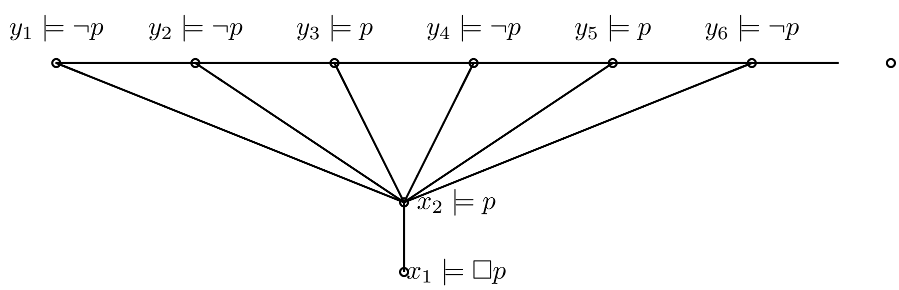

Diagram 1.

Let us define inductively a sequence of formulas A_i, $i \geq 1$ constructed from one variable p. Denote $\alpha := p \wedge \neg\Diamond\Box p$.

Definition 8.

$$\begin{aligned} A_1 &:= \neg p \wedge \Box\neg\alpha \\ A_2 &:= \neg p \wedge \neg A_1 \wedge \Diamond A_1 \\ A_3 &:= \alpha \wedge \Diamond A_2 \end{aligned}$$

For $n \geq 2$:

$$\begin{aligned} A_{2n} &:= \neg p \wedge \Diamond A_{2n-1} \wedge \neg A_{2n-2} \\ A_{2n+1} &:= \alpha \wedge \Diamond A_{2n} \wedge \neg A_{2n-1} \end{aligned}$$

Lemma 9. *For any $i \geq 1$ and for any $x \in W$ the following holds:*

$$x \models A_i \quad \textit{iff} \quad x = y_i$$

Proof. Let $Y_p := \{y_{2m+1}, m \geq 1\}$ and $Y_{\neg p} := \{y_1\} \cup \{y_{2m}, m \geq 1\}$. We see the point x_1 is the only point which validates $\Box p$. Analogously, the point x_2 is the only point such that: $x_2 \models \Diamond\Box p \wedge \neg\Box p$. We also see that $y_{2n+1} \models \alpha$ for $n \geq 1$ while for the others points this not hold. Now, we use induction on i.
Case $i = 1$. Obviously $y_1 \models A_1$. Suppose $x \models A_1$ for some arbitrary $x \in W$. Then $x \in Y_{\neg p}$. Because $x \models \Box\neg\alpha$ then there is no point y_{2m+1}, $m \geq 1$ being in such that xR_{2m+1}. Hence it must be $x = y_1$.
Case $i = 2$. Of course $y_2 \models A_2$. Conversely, suppose $x \models A_2$ for some $x \in W$. First, $x \in Y_{\neg p}$ and $x \neq y_1$ because $x \models \neg A_1$. Because $x \models \Diamond A_1$ then we conclude $x = y_2$.
Case i=3. Obviously $y_3 \models A_3$. Suppose $x \models A_3$ for some $x \in W$. Then $x \in Y_p$. Because $x \models \Diamond A_2$ then $x = y_3$.
Case $i = 2n$ for $n \geq 2$. The induction hypothesis is that the lemma holds for $i = 2n-1, 2n-2$. Of course $y_{2n} \models A_{2n}$. Let us suppose that $x \models A_{2n}$. Then we have $x \in Y_{\neg p}$. Since $x \models \Diamond A_{2n-1}$ then x must be y_{2n-2} or y_{2n}. From induction hypothesis and because $x \models \neg A_{2n-2}$ we conclude $x = y_{2n}$.
Case $i = 2n+1$ for $n \geq 2$ proceeds analogously to the above. $\Box$

M. Byrd in [1] proved that there are infinitely many non-equivalent formulas written in two variables in the logic $\mathbf{T_2}$. He strongly believed that the number of such formulas written in one variable is finite. His conjecture was disproved in 1981 by D. Makinson who first in [3] constructed an infinite sequence of non-equivalent formulas in $\mathbf{T_2}$ written in one variable. His construction was made for strong omnitemporal logic $\mathbf{B}(S4.3, S4)$ which is a supersystem of $\mathbf{T_2}$. By Lemma 9, we obtain another example of a such sequence.

Theorem 10. *The formulas $\{A_i\}$, $i \geq 1$ are non-equivalent in the logic $\mathbf{T_2}$.*

Proof. For any $k \neq l$ we have in our model $\mathfrak{M}$ that $y_k \not\models A_k \to A_l$ and $y_l \not\models A_l \to A_k$. Because $\mathfrak{M}$ is $\mathbf{T_2}$ model then we have $A_k \to A_l \notin \mathbf{T_2}$ and $A_l \to A_k \notin \mathbf{T_2}$. $\Box$

From inclusions (1) and Theorem 10 we immediately have:

Corollary 11. *For any $n \geq 2$ the logic $\mathbf{T_n}$ has infinitely many non-equivalent formulas written in one variable.*

3 The existence of a continuum of logics over $\mathbf{T_2}$

Y. Miyazaki in [4] and [5] proved that there is a continuum of normal modal logics over $\mathbf{T_2}$. In the first paper, he considered logics determined by the so-called wheel frame. In the second one, he showed the existence of continuum of orthologics, and then applied an embedding from orthologics to **KTB** logics. In this section we present another construction of continuum of logics over $\mathbf{T_2}$. We take advantage of the model $\mathfrak{M}$ from the previous section and construct a family of new models which characterizes a family of extensions of $\mathbf{T_2}$ logic. The family is obtained by adding new axioms written in one variable only.

Definition 12.

$$\mathfrak{M}_{\mathrm{i}} = \langle W_i, R_i, V_i \rangle, \quad i \geq 3,$$

where

$$W_i := \{x_1, x_2\} \cup \{y_1, ..., y_i\},$$
$$V_i(p) := \{x_1, x_2\} \cup \{y_{2m+1}, m \geq 1\}.$$

and the relation R_i is a restriction of the relation R from the model $\mathfrak{M}$ to the set W_i.

For example, in Diagram 2 we present the diagrams of models $\mathfrak{M}_3$ and $\mathfrak{M}_4$.

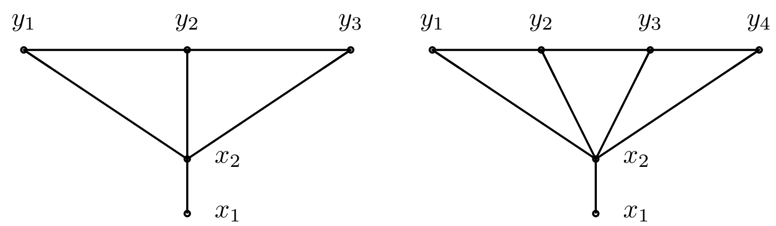

Diagram 2.

Let us consider another sequence of formulas built from the formulas A_i defined in Definition 8.

Definition 13.

$$B_i \quad := \quad (\Box p \wedge C_1 \wedge C_2 \wedge \bigvee_{k=3}^{i} C_k\,) \to \Diamond[\bigwedge_{k=1}^{i-1} \Diamond A_k \wedge \Diamond(A_i \wedge \neg\Diamond A_{i+1})],$$

for $i \geq 3$,
where

$$C_i \quad := \quad \Box p \to \Diamond^2 A_i, \textit{ for } i \geq 1.$$

Lemma 14. *For any $i \geq 3$ and for any $x \in W_i$ the following holds:*

$$(\mathfrak{M}_j, x) \models B_i \quad \textit{iff} \quad i = j.$$

Proof. First, we show that $(\mathfrak{M}_i, x) \models B_i$ for $i \geq 3$. From the definition of model $\mathfrak{M}_i$ we know that $(\mathfrak{M}_i, x) \models \Box p \quad$ iff $\quad x = x_1$.
Hence, for any $x \neq x_1$ we have $(\mathfrak{M}_i, x) \models B_i$. Now, we check he behavior of B_i at x_1. Because in $\mathfrak{M}_i$ there are exactly i points $y_1, y_2, ..., y_i$ then

$$(\mathfrak{M}_i, x_1) \models C_i \quad \text{for any} \quad i \geq 3.$$

Hence $(\mathfrak{M}_i, x_1) \models \Box p \wedge C_1 \wedge C_2 \wedge \bigvee_{k=3}^{i} C_k$. Let us notice that the point x_2 is the one such that:

$$(\mathfrak{M}_i, x_2) \models \bigwedge_{k=1}^{i-1} \Diamond A_k \wedge \Diamond(A_i \wedge \neg\Diamond A_{i+1}).$$

Hence $(\mathfrak{M}_i, x_1) \models \Diamond[\bigwedge_{k=1}^{i-1} \Diamond A_k \wedge \Diamond(A_i \wedge \neg\Diamond A_{i+1})]$ and $(\mathfrak{M}_i, x_1) \models B_i$.
To prove the reverse implication, suppose we have $i \neq j$. We show $(\mathfrak{M}_j, x) \not\models B_i$. Exactly, we show that $(\mathfrak{M}_j, x_1) \not\models B_i$. If $3 \leq j < i$ then

$$(\mathfrak{M}_j, x_1) \models \Box p \wedge C_1 \wedge C_2 \wedge \bigvee_{k=3}^{i} C_k$$

and

$$(\mathfrak{M}_j, x_2) \not\models \Diamond A_i.$$

The last fact involves that:

$$(\mathfrak{M}_j, x_1) \not\models \Diamond[\bigwedge_{k=1}^{i-1} \Diamond A_k \wedge \Diamond(A_i \wedge \neg\Diamond A_{i+1})],$$

what gives us $(\mathfrak{M}_j, x_1) \not\models B_i$.
Let us suppose that $3 \leq i < j$. Then we have

$$(\mathfrak{M}_j, x_1) \models \Diamond[\Diamond A_1 \wedge \Diamond A_2 \wedge ... \wedge \Diamond A_i \wedge \Diamond A_{i+1} \wedge .... \wedge \Diamond(A_j \wedge \neg\Diamond A_{j+1})].$$

Hence of course:

$$(\mathfrak{M}_j, x_1) \not\models \Diamond[\Diamond A_1 \wedge \Diamond A_2 \wedge ... \wedge \Diamond(A_i \wedge \neg\Diamond A_{i+1})].$$

Because $(\mathfrak{M}_j, x_1) \models \Box p \wedge C_1 \wedge C_2 \wedge \bigvee_{k=3}^{i} C_k$ then $(\mathfrak{M}_j, x_1) \not\models B_i$. $\Box$

Lemma 15. *For any substitution $h : \{p\} \to \mathcal{ML}_1$ and for any $x \in W_i$, $(\mathfrak{M}_i, x) \models h(B_i)$.*

Proof. We start with enumeration the all different modalities in $\mathbf{T_2}$. From [1] we have the following sequences of positive and negative modalities (from the strongest one to the weakest):

$$\Box^2 p \rightsquigarrow \Box p \rightsquigarrow \Diamond\Box p \rightsquigarrow \Diamond^2\Box p \rightsquigarrow \Box^2\Diamond p \rightsquigarrow \Box\Diamond p \rightsquigarrow \Diamond p \rightsquigarrow \Diamond^2 p \tag{10}$$

$$\Box^2\neg p \rightsquigarrow \Box\neg p \rightsquigarrow \Diamond\Box\neg p \rightsquigarrow \Diamond^2\Box\neg p \rightsquigarrow \Box^2\Diamond\neg p \rightsquigarrow \Box\Diamond\neg p \rightsquigarrow \Diamond\neg p \rightsquigarrow \Diamond^2\neg p \tag{11}$$

The model $\mathfrak{M}_i$ is fixed. First, we consider the following substitutions $h : \{p\} \to \{\Box^2 p, \Box p, \Box^2\neg p, \Box\neg p, \Diamond\Box\neg p, \Diamond^2\Box\neg p, \Box^2\Diamond\neg p\}$. They are such that for any $x \in W_i$ $(\mathfrak{M}_i, x) \not\models h(\Box p)$. Hence, for any $x \in W_i$, $(\mathfrak{M}_i, x) \models h(B_i)$.
Let us consider other substitutions: $h^* : \{p\} \to \{\Diamond^2\Box p, \Box^2\Diamond p, \Box\Diamond p, \Diamond p, \Diamond^2 p, \Diamond^2\neg p\}$. For any such h^* and for any $x \in W_i$ we have $(\mathfrak{M}_i, x) \models h^*(\Box p)$. But then for any $x \in W_i$, $(\mathfrak{M}_i, x) \not\models h^*(A_1)$ and furthermore $(\mathfrak{M}_i, x) \models h^*(B_i)$.
Two substitutions do not fit to the above ones: $h_1(p) = \Diamond\Box p$ and $h_2(p) = \Diamond\neg p$. For these substitutions we have for any $x \in W_i$, $(\mathfrak{M}_i, x) \not\models h_k(A_2)$, for $k = 1, 2$ and hence $(\mathfrak{M}_i, x) \models h_k(B_i)$ for $k = 1, 2$.
Now, we should consider complex formulas. It is easy to observe that negations of formulas from the sequence (10) give the formulas from (11) and inversely. Consideration of conjunctions and disjunctions between formulas only from (10)(or only

from (11)) gives nothing new. Suppose, we substitute p for conjunction such that $h'(p) = h(p) \wedge g(p)$ where $g \in \{h^*, h_1, h_2\}$. Then for any $x \in W_i$ we obtain $(\mathfrak{M}_i, x) \not\models h'(\Box p)$ and also $(\mathfrak{M}_i, x) \models h'(B_i)$. On the other side if we take two substitutions into the set $\{\Diamond^2\Box p,\, \Box^2\Diamond p,\, \Box\Diamond p,\, \Diamond p,\, \Diamond^2 p,\, \Diamond^2\neg p\}$ then the formula being their conjunction behaves exactly in the same way as it parts and the thesis holds. Suppose we have the following substitution: $h''(p) = h_1(p) \wedge h_2(p)$. We check that $\forall_{x \neq x_1}(\mathfrak{M}_i, x) \not\models h_1(\Box p)$ and $\forall_{x \notin \{y_1,...,y_i\}}(\mathfrak{M}_i, x) \not\models h_2(\Box p)$. Hence for conjunction we have: $\forall_{x \in W_i}(\mathfrak{M}_i, x) \not\models h''(\Box p)$, what immediately gives us $(\mathfrak{M}_i, x) \models h''(B_i)$
Analogously, we should consider the case of disjunction. In this case we have to start with the analysis of the points from W_i at which $h(p)$ (not $h(\Box p)$) is true (for different h).
As we see the substitution the variable p for conjunction (or disjunction) of different modalities leads to obtaining some trivial cases (such that $h(\Box p)$ or $h(A_1)$ is everywhere in model false). If we build a conjunction (or disjunction) from some modality and formula p (or $\neg p$), then we do not obtain a new situation. Then we may conclude, that any substitution $h : \{p\} \to \mathcal{ML}_1$ leads to the considered before cases. □

Lemma 16. *Let $\beta \in \mathbf{T_2} \oplus B_i$. Then for any substitution $h : \{p\} \to \mathcal{ML}_1$ and for any $x \in W_i$, $(\mathfrak{M}_i, x) \models h(\beta)$.*

Proof. Suppose that we have the proof $\beta_1, ...\beta_k$ of the fact that $\beta \in \mathbf{T_2} \oplus B_i$. Then $\beta_k = \beta$. We use induction with respect to length k of that proof. Case 1. $k = 1$. Then $\beta \in T_2 \cup \{B_i\}$. From Lemma 15 it is clear that the thesis holds.
Case 2. Let us suppose that for the length $\leq k$ thesis holds. We show that for the k-th formula in the proof it also holds.
Case 2a. If $\beta \in T_2 \cup \{B_i\}$, then it is obvious. Case 2b. If β was obtained by derivation, then there are two following formulas $\beta_p = \beta_t \to \beta_k$, β_t, $p, t < k$. From inductive hypothesis we know that: for any substitution h and for any $x \in W_i$ $(\mathfrak{M}_i, x) \models h(\beta_t \to \beta_k)$ and $(\mathfrak{M}_i, x) \models h(\beta_t)$. Because $h(\beta_t \to \beta_k) = h(\beta_t) \to h(\beta_k)$ then by derivation we have: $(\mathfrak{M}_i, x) \models h(\beta_k)$.
Case 2c. If β is obtained generalization, then $\beta_k = \Box\beta_p$ for some $p < k$. Again from inductive hypothesis: $(\mathfrak{M}_i, x) \models h(\beta_p)$. Because x is any point from W_i then also: $(\mathfrak{M}_i, x) \models h(\Box\beta_p)$.
Case 2d. Suppose β is obtained by substitution. Then for some substitution g of some formula β_p, $p < k$ we have $g(\beta_p) = \beta_k$. But then the superposition $h \circ g$ is also the substitution and from inductive hypothesis we immediately have $(\mathfrak{M}_i, x) \models h \circ g(\beta)$.

□

Lemma 17. *$B_j \notin \mathbf{T_2} \oplus B_i$ for any $i, j \geq 3$ such $i \neq j$.*

Proof. Suppose on contrary that $B_j \in \mathbf{T_2} \oplus B_i$ for $i \neq j$. From Lemma 16, for substitution being identity we immediately have that for any $x \in W_i$, $(\mathfrak{M}_i, x) \models id(B_j)$. This is a contradiction with Lemma 14. □

Let us define a new family of models $\mathfrak{M}_X$, $X \in \omega$ and $1, 2 \notin X$.

Definition 18.

$$\mathfrak{M}_X = \langle W_X, R_X, V_X \rangle,$$

where

$$W_X := \{x_1, x_2\} \cup \bigcup_{i \in X} \{y_1^i, ..., y_i^i\},$$

$$V_X := \{x_1, x_2\} \cup \{y_{2m+1}^i,\ m \geq 1,\ i \in X\}.$$

and the relation R_X is a restriction of the relation R from the model $\mathfrak{M}$ to the sets $\{y_1^i, ..., y_i^i\}$, where $i \in X$.

For example, in Diagram 3 we present model $\mathfrak{M}_X$ where $3, 5, 7 \in X$.

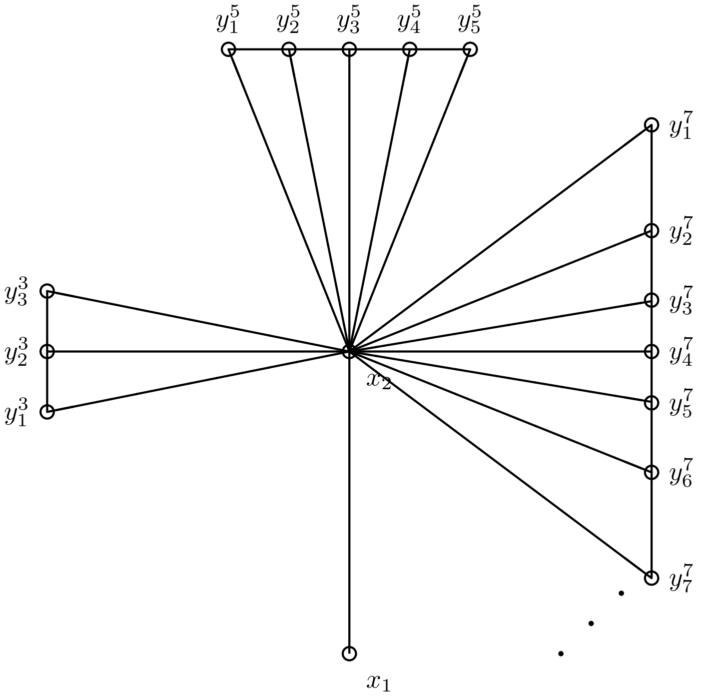

Diagram 3.

Now, we are able to generalize Lemmas 14-17 in the following way:

Corollary 19. *For any $i \geq 3$ and for any $x \in W_X$ the following holds:*

$$(\mathfrak{M}_X, x) \models B_i \quad \textit{iff} \quad i \in X$$

Corollary 20. *For any substitution $h : \{p\} \to \mathcal{ML}_1$ and for any $x \in W_X$, $(\mathfrak{M}_X, x) \models h(B_i)$.*

Corollary 21. *Let $\beta \in \mathbf{T_2} \oplus \{B_i,\, i \in X\}$. Then for any substitution $h : \{p\} \to \mathcal{ML}_1$ and for any $x \in W_X$, $(\mathfrak{M}_X, x) \models h(\beta)$.*

Corollary 22. *For any $j \notin X$ formula $B_j \notin \mathbf{T_2} \oplus \{B_i,\, i \in X\}$.*

Theorem 23. *There is a continuum of logics in $NEXT(\mathbf{T_2})$, where every member is axiomatized by formulas on one variable.*

Proof. Let $X, Y \subseteq \omega$, ($X \neq Y$, $1, 2 \notin X$ and $1, 2 \notin Y$). Consider logics: $\mathbf{T_2} \oplus \{B_i,\, i \in X\}$ and $\mathbf{T_2} \oplus \{B_j,\, j \in Y\}$. From Corollary 22 we know, that they are different from each other. $\square$

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