On asymptotic divergency]On asymptotic divergency in equivalential logics Zofia Kostrzycka] Zofia Kostrzycka,

University of Technology,
Luboszycka 3, 45-036 Opole, Poland,
E-mail zkostrz@po.opole.pl

[

[

March 12, 2006

Abstract

In this paper the equivalential reducts of classical and intuitionistic logics over language with two propositional variables are characterized. Next, the size of the fraction of tautologies of these logics against all formulas are investigated. To do that quite non-logical methods are used.

1 Introduction

For the given logical calculus we investigate the proportion of the number of true formulas of a certain length n to the number of all formulas of such length. We are especially interested in asymptotic behavior of this fraction when n tends to infinity. If the limit exists it is represented by a real number between 0 and 1 which we call *the density of truth* for the investigated logic.
Let L be some logical calculus. Let $|T_n|$ be a number of tautologies of length n of that calculus and $|F_n|$ be a number of all formulas of that length. We define the density $\mu(L)$ as:

$$\mu(L) = \lim_{n\to\infty} \frac{|T_n|}{|F_n|}$$

This paper is continuation of quantitative research in different logics. Our interest concerns mostly the classical and intuitionistic logic. Until now, the *density of truth* for classical (and intuitionistic) logic of implication of one and two variables are known (see [Moczurad, Tyszkiewicz and Zaionc 2000], [Kostrzycka 2003]) as well as the *density* of implicational-negational fragments of these calculi with one variable (see [Zaionc 2004], [Kostrzycka and Zaionc 2004]).
In case of equivalential reducts of these logics the problem of existence of the *density of truth* is more complicated. It has been proved in [Matecki 2005] that equivalential reduct of classical logic is not asymptotically convergent, what means that the *density of truth* (as a limit) does not exist. However it is possible to count limit superior and limit inferior for the fractions of tautologies, which seems to be worth counting. In this paper we consider the case of equivalential reduct of intuitionistic logic with two variables and compare the obtained result with the one for classical logic.

2 Equivalential reduct of intuitionistic logic INT_E

The equivalential reducts INT_E of intuitionistic propositional logic was widely investigated in its algebraic counterpart - equivalential algebras (for example see [Słomczyńska 1996], [Wroński 1993]).

The class E of equivalential algebras is equationally definable by the following identities:

$$(x \leftrightarrow x) \leftrightarrow y = y$$
$$((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z = (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z)$$
$$(x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) = x \leftrightarrow y$$

The class E forms a variety and is locally finite. The cardinality of the n-generated free algebra $F_E(n)$ is known only for $n = 1, 2, 3$ and is equal to 2, 9, 4415434, respectively; see [Wroński 1993].
The n-generated free algebra $F_E(n)$ is isomorphic to the Lindenbaum algebra $AL(E(n))$ over language consisted of n variables. We will consider the algebra $AL(E(n))$ with $n = 2$.
Let us define the language $\mathcal{F}^{\{\leftrightarrow\}}$ consisting of equivalential formulas with two propositional variables p and q:

Definition 1.

$$p,\ q \in \mathcal{F}^{\{\leftrightarrow\}}$$
$$\phi \leftrightarrow \psi \in \mathcal{F}^{\{\leftrightarrow\}} \quad \textit{iff} \ \ \phi \in \mathcal{F}^{\{\leftrightarrow\}} \ \ \textit{and} \ \ \psi \in \mathcal{F}^{\{\leftrightarrow\}}$$

In the set $\mathcal{F}^{\{\leftrightarrow\}}$ we can introduce an equivalence relation $\equiv$ in the conventional way:

$$\alpha \equiv \beta \ \text{ iff } \ \alpha \leftrightarrow \beta \in INT_E \tag{1}$$

The equivalence relation $\equiv$ is also a congruence relation and the quotient algebra $\mathcal{F}^{\{\leftrightarrow\}}/_{\equiv}$ is called the Lindembaum algebra of the logic INT_E. We denote it by:

$$AL(INT_E) = \mathcal{F}^{\{\leftrightarrow\}}/_{\equiv} \tag{2}$$

From [Wroński 1993] we have

Theorem 2. *The Lindembaum algebra $AL(INT_E)$ consists of the following 9 equivalence classes:*

$$\begin{aligned}
I &= [p]_{\equiv} \\
II &= [q]_{\equiv} \\
III &= [p \leftrightarrow q]_{\equiv} \\
IV &= [(p \leftrightarrow q) \leftrightarrow q]_{\equiv} \\
V &= [(p \leftrightarrow q) \leftrightarrow p]_{\equiv} \\
VI &= [((p \leftrightarrow q) \leftrightarrow q) \leftrightarrow p]_{\equiv} \\
VII &= [((p \leftrightarrow q) \leftrightarrow p) \leftrightarrow q]_{\equiv} \\
VIII &= [(((p \leftrightarrow q) \leftrightarrow q) \leftrightarrow p) \leftrightarrow (((p \leftrightarrow q) \leftrightarrow p) \leftrightarrow q)]_{\equiv} \\
T &= [p \rightarrow p]_{\equiv}
\end{aligned}$$

The behavior of the operator $\leftrightarrow$ is presented in Table 1. Diagram of the algebra $AL(INT_E)$ is presented in Figure 1.
Let us remind the notion of the equivalential filter.

Definition 3. *Let A be an equalential algebra. By a filter of A we mean a non-empty subset F of A such that for all $a, x \in A$:*
(i) if $a \in F$, then $(a \leftrightarrow x) \leftrightarrow x \in F$,
(ii) if $a \in F$ and $a \leftrightarrow b \in F$, then $b \in F$.

Figure 1: Diagram of the algebra $AL(INT_E)$

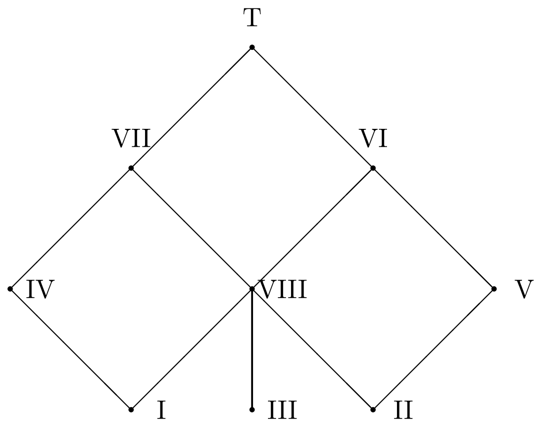
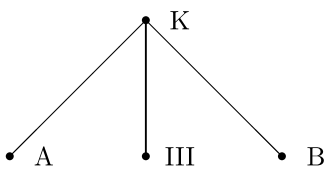


Table 1: The truth-table of algebra $AL(INT_E)$

$\leftrightarrow$	I	II	III	IV	V	VI	VII	VIII	T
I	T	III	V	VI	III	IV	I	IV	I
II	III	T	IV	III	VII	II	V	V	II
III	V	IV	T	V	IV	III	III	III	III
IV	VI	III	V	T	III	I	IV	I	IV
V	III	VII	IV	III	T	V	II	II	V
VI	IV	II	III	I	V	T	VIII	VII	VI
VII	I	V	III	IV	II	VIII	T	VI	VII
VIII	IV	V	III	I	II	VII	VI	T	VIII
T	I	II	III	IV	V	VI	VII	VIII	T

Figure 2: Diagram of the algebra $AL(INT_E)/_K$



where $A = I \cup IV, \quad B = II \cup V$

Table 2: The truth-table of the algebra $AL(INT_E)/_K$

$\leftrightarrow$	A	B	III	K
A	K	III	B	A
B	III	K	A	B
III	B	A	K	III
K	A	B	III	K

Let us observe that there is only one proper equivalential filter in the algebra $AL(INT_E)$. This is the set $K = \{T, VI, VII, VIII\}$. The new quotient algebra $AL(INT_E)/_K$ is presented in Figure 2 and the operation of $\leftrightarrow$ in it is characterized in Table 2.

Lemma 4. *The algebra $AL(INT_E)/_K$ is equal to the Lindenbaum algebra $AL(CL_E)$ of the classical propositional logic of equivalence with two variables.*

Proof. For the proof we split the set of all formulas into some classes according to formulas' behavior on all possible evaluations. Since we have classical formulas built with exactly two propositional variables we can evaluate them by four valuations: ν_1 associating 0 to p and 0 to q, ν_2 associating 0 to p and 1 to q, ν_3 associating 1 to p and 0 to q and ν_4 associating 1 to p and 1 to q .
For any $i, j, k, l \in \{0, 1\}$ by $F^{i,j,k,l}$ we mean the set of formulas ϕ from $\mathcal{F}^{\{\leftrightarrow\}}$ such that $\nu_1(\phi) = i$, $\nu_2(\phi) = j$, $\nu_3(\phi) = k$ and $\nu_4(\phi) = l$.
There are exactly four classes:

$$F^{0,0,1,1}$$
$$F^{0,1,0,1}$$
$$F^{1,0,0,1}$$
$$F^{1,1,1,1}$$

The four classes $F^{i,j,k,l}$ are ordered by assuming that $F^{i,j,k,l} \leq F^{i',j',k',l'}$ iff $i \leq i'$, $j \leq j'$, $k \leq k'$ and $l \leq l'$. On the classes $F^{i,j,k,l}$ we can establish an operation of equivalence $\leftrightarrow$ by $F^{i,j,k,l} \leftrightarrow F^{i',j',k',l'} = F^{i\Leftrightarrow i', j\Leftrightarrow j', k\Leftrightarrow k', l\Leftrightarrow l'}$ where $\Leftrightarrow$ stands for the classical equivalence defined on the set $\{0, 1\}$. By the construction above we get the four element lattice, exactly the same as we can see in Figure 2. We have the following equalities:

$$\begin{aligned} A &= F^{0,0,1,1} \\ B &= F^{0,1,0,1} \\ III &= F^{1,0,0,1} \\ K &= F^{1,1,1,1} \end{aligned}$$

Table 2 is identical with the one defined by operations $\Leftrightarrow$ described above. As we can see the class K is the class of classical tautologies.

□

3 Counting formulas

In this section we establish some properties of numbers characterizing the amount of formulas in different classes defined in our language. First, set up the way of measuring length of formulas.

Definition 5. *By $|\phi|$ we mean the length of formula ϕ which is the total number of occurrences of propositional variables in the formula, excluding the equivalence sign and parenthesis. Formally*

$$\begin{aligned} |p| &= 1 \\ |q| &= 1 \\ |\phi \leftrightarrow \psi| &= |\phi| + |\psi| \end{aligned}$$

We will consider the set $F_n \subseteq \mathcal{F}^{\{\leftrightarrow\}}$ of all formulas of the length n. Its appropriate subclasses were defined as follows:

Definition 6.

$$Y_n \quad = \quad F_n \cap Y \;\; \textit{for any} \;\; Y \in 2^{AL(INT_E)}$$

The number of formulas in F_n is finite for any $n \in N$ and will be denoted by $|F_n|$. Consequently any subset from the class $2^{AL(INT_E)}$ is also finite for any $n \in \mathbb{N}$.

Definition 7. *By $|Y_n|$ we mean the number of formulas from the class Y_n.*

From Definition 1 we see that any formula from $\mathcal{F}^{\{\leftrightarrow\}}$ may be interpreted as a binary planar tree with the internal nodes labelled by the operator $\leftrightarrow$ and the external nodes by the propositional variables p and q. Then we have immediately:

Lemma 8. *The numbers $|F_n|$ are given by the following recursion:*

$$|F_0| \quad = \quad 0, \;\; |F_1| = 2, \tag{3}$$

$$|F_n| \quad = \quad \sum_{i=1}^{n-1} |F_i||F_{n-i}| \tag{4}$$

Proof. Obvious. □

The numbers $|F_n|$ are in close connection with the Catalan numbers C_n. The well known non-recursive formula for C_n is the following:

$$C_n = \frac{1}{n}\binom{2n-2}{n-1} \tag{5}$$

It is easy to observe that $|F_n| = 2^n \cdot C_n$.

4 Generating functions

The main tool for dealing with asymptotic of sequences of numbers are *generating functions*; see for example [Wilf 1994],[Flajolet and Sedgewick 2001]. Suppose, we have a system of non-linear equations $\overrightarrow{y_j} = \Phi_j(z, y_1, ...y_m)$ for $1 \leq j \leq m$, where any $y_j = \sum_{n=0}^{\infty} a_j z^n$. The following result known as Drmota-Lalley-Woods theorem is of great importance in both cases of solving the system explicitly or implicitly; see [Flajolet and Sedgewick 2001], Thm. 8.13, p.71:

Theorem 9. *Consider a nonlinear polynomial system $\overrightarrow{y} = \Phi(\overrightarrow{y})$ that is a-proper, a-positive and a-irreducible. In that case, all component solutions y_i have the same radius of convergence $\rho < \infty$. Then, there exist functions h_j analytic at the origin such that*

$$y_j = h_j(\sqrt{1 - z/\rho}), \quad (z \to \rho^-). \tag{6}$$

In addition, all other dominant singularities are of the form $\rho\omega$ with ω being a root of unity. If furthermore the system is a-aperiodic then all y_j have ρ as unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form:

$$[z^n]y_j(z) \sim \rho^{-n} \left(\sum_{k \geq 1} d_k n^{-1-k/2} \right). \tag{7}$$

The expression from the right side of (7) may be transformed by the so called transfer lemma [Flajolet and Odlyzko 1990] into formula defining the value of the coefficients $[z^n]y_j(z)$ explicitly. So, the a-aperiodicity of a system of equations is a very desirable property. Unfortunately, in our considerations this property does not take place and we need another formula to approximate the coefficients $[z^n]y_j(z)$. We take advantage of the Szegö Theorem from [Szegö 1975] [Thm. 8.4], see also [Wilf 1994] [Thm. 5.3.2].

Lemma 10. *[Szegö] Let $v(z)$ be analytic in $|z| < 1$ with a finite number of singularities $e^{i\varphi^(k)}$, $k = 1, ..., s$ at the circle $|z| = 1$. Suppose that in the neighborhood of each $e^{i\varphi^(k)}$, $v(z)$ has the expansion of the form*

$$v(z) = \sum_{p\geq 0} v_p^{(k)}(1 - ze^{-i\varphi^{(k)}})^{a^{(k)}+pb^{(k)}},$$

where $a^{(k)} \in C$ and $b^{(k)} > 0$ and the branch chosen above for the expansion equals $v(0)$ for $z = 0$. Then

$$[z^n]\{v(z)\} = \sum_{k=1}^{s}\sum_{p=0}^{\xi(q)} v_p^{(k)}\binom{a^{(k)} + pb^{(k)}}{n}(-e^{i\varphi^{(k)}})^n + O(n^{-q}).$$

with

$$\xi(q) = \max_{k=1,\dots,s}[(1/b^{(k)})(q - \Re(a^{(k)}) - 1)]$$

.

From the first part of Theorem 9 we see that in the case when generating functions form a-proper, a-positive and a-irreducible system of equations, $a^{(k)} = 0$ and $b^{(k)} = 1/2$. Further simplification depends on the numbers of singularities. In our application we will only have one or two singularities. In both cases we will be satisfied with error bound $O(n^{-2})$. Then $\xi(q) = 2$. Moreover, $\binom{0}{n} = \binom{1}{n} = 0$ for $n > 1$. Additionally, for the purpose of needed calculations we admit the singularities are situated at the circle $|z| = \rho$. If there is only one singularity $z_0 = \rho$ then we have:

Lemma 11. *Let $v(z)$ be analytic in $|z| < \rho$ with $z = \rho$ being the only singularity at the circle $|z| = \rho$. If $v(z)$ in the vicinity of $z = \rho$ has an expansion of the form*

$$v(z) = \sum_{p\geq 0} v_p(1 - z/\rho)^{\frac{p}{2}}, \tag{8}$$

where the branch chosen above for the expansion equals $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = \left(v_1\binom{1/2}{n}(-1)^n + O(n^{-2})\right)\cdot\rho^{-n}. \tag{9}$$

Proof. Function $\widehat{v}(z) = v(z\cdot\rho) = \sum_{p\geq 0} v_p(1 - z)^{\frac{p}{2}}$. From the Szegö Lemma we have:

$$[z^n]\{\widehat{v}(z)\} = \left(\widehat{v}_1\binom{1/2}{n}(-1)^n + O(n^{-2})\right) \tag{10}$$

From definition $v(z) = \sum_{n=0}^{\infty} a_j z^n$ we have $\widehat{v}(z) = \sum_{n=0}^{\infty} a_j(z\rho)^n$. Hence

$$[z^n]\{v(z)\} = [z^n]\{\widehat{v}(z)\}\cdot\rho^{-n}. \tag{11}$$

□

For two singularities $z = \rho$ and $z = -\rho$ we have:

Lemma 12. *Let $v(z)$ be analytic in $|z| < \rho$ with $z_1 = \rho$ and $z_2 = -\rho$ being the only singularities at the circle $|z| = \rho$. If $v(z)$ in the neighborhood of $z = \rho$ has expansion of the form*

$$v(z) = \sum_{p \geq 0} v_p^{(1)} (1 - z/\rho)^{\frac{p}{2}}, \tag{12}$$

where the branch chosen above for the expansion equals to $v(0)$ for $z = 0$, and again if $v(z)$ in the neighborhood of $z = -\rho$ has expansion of the form

$$v(z) = \sum_{p \geq 0} v_p^{(2)} (1 + z/\rho)^{\frac{p}{2}}, \tag{13}$$

where the branch chosen above for the expansion equals to $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = \left(\left(v_1^{(1)} + v_1^{(2)} \cdot (-1)^n \right) (-1)^n \binom{1/2}{n} + O(n^{-2}) \right) \rho^{-n}. \tag{14}$$

Proof. Analogous to the proof of Lemma 11. □

For technical reasons we will need to know the rate of grow of the expression $\binom{1/2}{n}(-1)^n$ which appears at formula (14).

Lemma 13.

$$\binom{1/2}{n}(-1)^{n+1} = O(n^{-3/2})$$

Proof. It can be obtained by the Stirling approximation formula (see [Robbins 1955] for details, consult also lemma 7.5 page 589 at [Moczurad, Tyszkiewicz and Zaionc 2000]).

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}. \tag{15}$$

□

From Lemmas 11 - 13 we obtain:

Lemma 14. *Suppose function $w(z)$ satisfies assumptions of Lemma 12 and function $v(z)$ satisfies assumptions of Lemma 11. Then the limit of $\frac{[z^n]\{w(z)\}}{[z^n]\{v(z)\}}$ is given by the formula:*

$$\lim_{n\to\infty} \frac{[z^n]\{w(z)\}}{[z^n]\{v(z)\}} = \lim_{n\to\infty} \frac{w_1^{(1)} + w_1^{(2)} \cdot (-1)^n}{v_1}$$

Proof. Applying the main formulas from the simplified versions of the Szegö Lemma (Lemma 11 and Lemma 12) and the equation from Lemma 13:

$$\begin{aligned}
\lim_{n\to\infty} \frac{[z^n]\{w(z)\}}{[z^n]\{v(z)\}} &= \lim_{n\to\infty} \frac{\left(\left(w_1^{(1)} + w_1^{(2)} \cdot (-1)^n \right) \cdot (-1)^n \binom{1/2}{n} + O(n^{-2}) \right) \rho^{-n}}{\left(v_1 \binom{1/2}{n} (-1)^n + O(n^{-2}) \right) \rho^{-n}} \\
&= \lim_{n\to\infty} \frac{\left(w_1^{(1)} + w_1^{(2)} \cdot (-1)^n \right) O(n^{-3/2}) + O(n^{-2})}{v_1 O(n^{-3/2}) + O(n^{-2})} \\
&= \lim_{n\to\infty} \frac{w_1^{(1)} + w_1^{(2)} \cdot (-1)^n}{v_1}.
\end{aligned}$$

□

5 Calculating generating functions

In this section we are going to determine generating functions for the classes of all equivalential formulas, equivalential classical tautologies and equivalential intuitionistic tautologies. A cursory analysis of Table 2 (and Table 1 as well) give an important information about the way of building formulas from each class. For example, to the class A belong formulas p, $(p \leftrightarrow q) \leftrightarrow q$, $q \leftrightarrow (p \leftrightarrow q)$, $(q \leftrightarrow p) \leftrightarrow q$, $q \leftrightarrow (q \leftrightarrow p)$ and so on. We see that some formula belongs to A if it is built from even number of variable q and odd number of variable p or only from odd number of variable p. That means the class A consists of formulas of the odd length only. Hence we have the following power series for the generating function f_A:

$$f_A(z) = z + 8z^3 + 224z^5 + ... \tag{16}$$

Analogously, the expansion of generating function for tautologies K is as follows:

$$f_K(z) = 2z^2 + 40z^4 + 1344z^6 + ... \tag{17}$$

As wee see each equivalential classical tautology has even length. Because $T \subset K$ then the same holds for the class T of equivalential intuitionistic tautologies. In such situation, counting the density $\mu(CL_E)$ and $\mu(INT_E)$ we will limit our interest to the formulas of even length. But even then we have to determine the needed generating functions f_K, f_T and f.
First, let us notice that the generating function f for numbers $|F_n|$ is the one for Catalan numbers with variable $2z$:

Lemma 15.

$$f(z) = \frac{1}{2} - \frac{\sqrt{1-8z}}{2} \tag{18}$$

Proof. Obvious. □

From Table 2 we obtain the following system of equations:

$$(*) \begin{cases} f_A &= 2(f_A f_K + f_B f_{III}) + z \\ f_B &= 2 f_A f_{III} + z \\ f_{III} &= 2(f_A f_B + f_{III} f_K) \\ f_K &= f_A^2 + f_B^2 + f_{III}^2 + f_K^2 \end{cases}$$

where the functions f_A, f_B, f_{III}, f_K are the generating functions for the numbers $|A_n|$, $|B_n|$, $|III_n|$, $|K_n|$ respectively.
The system is a-positive, a-proper and a-irreducible and is not a-aperiodic. The last fact results for example from the expansion 17.
To determine the function f_K from the above system, we simplify it significantly. From Figure 2 we observe that

Lemma 16. *The following equalities hold between the appropriate generating functions:*

$$f_A = f_B, \tag{19}$$
$$f_{III} = f_K. \tag{20}$$

Proof. The first equality follows from symmetry of diagram of the algebra $AL(CL_E)$ presented in Figure 2. It is easy to notice that in any formula $\alpha \in A_i$ we may exchange variables p on q and q on p and obtain the appropriate formula from B_i. To prove the equality (20)let us observe that applying (19) to the system $(*)$ we obtain:

$$f_{III} = 2(f_A^2 + f_{III} f_K), \tag{21}$$
$$f_K = 2f_A^2 + f_{III}^2 + f_K^2. \tag{22}$$

By reduction of f_A we have:

$$f_K = f_{III} - 2f_{III} f_K + f_{III}^2 + f_K^2. \tag{23}$$

By solving it with the boundary condition $f_K(0) = 0$ we obtain $f_K = f_{III}$. □

Hence we have much simpler system of equations:

$$(**) \begin{cases} f_A &= 4f_A f_K + z \\ f_B &= f_A \\ f_{III} &= f_K \\ f_K &= 2(f_A^2 + f_K^2) \end{cases}$$

Lemma 17. *The generating function f_A is the following:*

$$f_A(z) = \frac{\sqrt{1+8z} - \sqrt{1-8z}}{8}. \tag{24}$$

Proof. From disjointness of classes A, B, III and K and from (19) and (20) we obtain $2f_K = f - 2f_A$. Applying it to the first equation from the system $(**)$, we have:

$$f_A(z) = (f(z) - 2f_A(z))f_A(z) + z \tag{25}$$

After a suitable simplification we get the following quadratic equation

$$4f_A^2 + f_A(1-f) - z = 0. \tag{26}$$

By solving it with the boundary condition $f_A(0) = 0$ we have $f_A = \frac{f-1+\sqrt{(1-f)^2+16z}}{8}$ and after replacing f with (18) we get (24).

□

From the equality $2f_K = f - 2f_A$ and from (24) we get:

Corollary 18. *The generating function f_K for the numbers $|K_n|$ is*

$$f_K(z) = \frac{2 - \sqrt{1+8z} - \sqrt{1-8z}}{8}. \tag{27}$$

Now, we are ready to analyze Table 1 characterizing the algebra $AL(INT_E)$. The appropriate system of equations consists of nine equations written in terms of functions $f_I - f_{VIII}$ and f_T. We do not write it explicitly; it is enough to mention the system again is a-positive, a-proper and a-irreducible and is not a-aperiodic. We are able to solve the system by taking the advantage of the way of obtaining the algebra $AL(INT_E)/_K = AL(CL_E)$:

Observation 19. *The following equalities between the appropriate generating functions:*

$$f_I + f_{IV} = f_{II} + f_V = f_A, \quad (28)$$
$$f_{VI} + f_{VII} + f_{VIII} + f_T = f_K. \quad (29)$$

From Figure 1 (and the symmetry of that diagram) we have:

Observation 20. *The following equalities hold between the appropriate pairs of generating functions:*

$$f_I = f_{II}, \quad (30)$$
$$f_{IV} = f_V, \quad (31)$$
$$f_{VI} = f_{VII}. \quad (32)$$

Further analysis of Table 1 gives us:

Lemma 21. *The generating function f_{VI} for the numbers $|VI_n|$ is*

$$f_{VI} = \frac{1}{32}(S - 2 - A - B) \quad (33)$$

where

$$A = \sqrt{1-8z} \quad \textit{and} \quad B = \sqrt{1+8z},$$
$$S = \sqrt{2 - 2AB + 8\sqrt{1 + A + B + AB - 12(A-B)z - 144z^2}}.$$

Proof. From Table 1 we obtain the following equalities between appropriate generating functions:

$$f_V = 2(f_I f_{III} + f_{II}(f_{VII} + f_{VIII}) + f_{III} f_{IV} + f_V(f_{VI} + f_T)), \quad (34)$$
$$f_{VI} = 2(f_I f_{IV} + f_{VI} f_T + f_{VII} f_{VIII}), \quad (35)$$
$$f_{VIII} = 2(f_{VI} f_{VII} + f_{VIII} f_T). \quad (36)$$

Application of (32) and (29) to (36) gives us

$$f_{VIII} = 2(f_{VI}^2 + f_{VIII}(f_K - f_{VIII} - 2f_{VI})). \quad (37)$$

By applying (28), (31) and (29) to (35) we get

$$f_{VI} = 2((f_A - f_V)f_V + f_{VI}(f_K - 2f_{VI})) \quad (38)$$

and by applying (30) - (29) to (34) and from $f_{III} = f_K$ we get after simplification

$$f_V = 2(f_A f_K + f_A(f_{VI} + f_{VIII}) + f_V(f_K - 2f_{VIII} - 2f_{VI})). \quad (39)$$

Let us notice the equations (39), (38) and (37) form a system of three equations with the unknown functions f_V, f_{VI} and f_{VIII}.
By reduction we get

$$f_V = \frac{f_A(-1 - 4f_K + \sqrt{16f_{VI}^2 + (1 + 4f_{VI} - 2f_K)^2} + 2f_K)}{2\sqrt{16f_{VI}^2 + (1 + 4f_{VI} - 2f_K)^2}}, \quad (40)$$
$$f_{VIII} = \frac{1}{4}\left(-1 - 4f_{VI} + \sqrt{16f_{VI}^2 + (1 + 4f_{VI} - 2f_K)^2} + 2f_K\right). \quad (41)$$

After substitution (40) to the equation (38) we obtain a four-degree equation with the unknown function f_{VI}. To solve it we had to intensively use *Mathematica* package and from four solutions we chose one fulfilling the boundary condition $f_{VI}(0) = 0$. After intensive simplification we get finally (33).

□

From Lemma 21 and from (41) we get

Corollary 22. *The generating function f_{VIII} for the numbers $|VIII_n|$ is the following:*

$$\begin{aligned} f_{VIII} &= \frac{1}{32}\Big(\sqrt{2}\sqrt{4+A^2+4B+B^2+2A(2+B)+S^2} \\ &\quad -2-A-B-S\Big), \end{aligned} \tag{42}$$

where

$$\begin{aligned} A &= \sqrt{1-8z} \quad \textit{and} \quad B=\sqrt{1+8z}, \\ S &= \sqrt{2-2AB+8\sqrt{1+A+B+AB-12(A-B)z-144z^2}}. \end{aligned}$$

And finally

Corollary 23. *The generating function f_T for the numbers of intuitionistic tautologies $|T_n|$ is as follows:*

$$f_T = \frac{1}{32}\Big(14-A-B-S-\sqrt{2}\sqrt{4+A^2+B(4+B)+2A(2+B)+S^2}\Big) \tag{43}$$

where

$$\begin{aligned} A &= \sqrt{1-8z} \quad \textit{and} \quad B=\sqrt{1+8z}, \\ S &= \sqrt{2-2AB+8\sqrt{1+A+B+AB-12(A-B)z-144z^2}}. \end{aligned}$$

6 Counting asymptotic densities

In this section we are going to expand the considered generating functions around their singularities. We will consider functions f, f_K and f_T.

Lemma 24. *$z_1 = \frac{1}{8}$ is the only singularity of the function f located in $|z| \le \frac{1}{8}$.*

Proof. Obvious. □

Lemma 25. *$z_1 = \frac{1}{8}$ and $z_2 = -\frac{1}{8}$ are the only singularities of f_K and f_T located in $|z| \le \frac{1}{8}$.*

Proof. It is easy to observe the function f_K have two singularities $z_1 = \frac{1}{8}$ and $z_2 = -\frac{1}{8}$. Because $f_{III} = f_K$ then f_{III} has the same singularities as f_K, only. Because f_{III} appears in the system of equations corresponding with Table 1, then from Theorem 9 we conclude that other functions from the system also have only the two singularities. This concerns also the function f_T. □

Theorem 26. *Expansion of function f in a neighborhood of $z = \frac{1}{8}$ is as follows:*

$$f(z) = f_0 + f_1\sqrt{1-8z} + ...$$

where

$$f_0 = \frac{1}{2}, \quad f_1 = -4$$

Proof. From Theorem 9 we conclude existence the function h_j being analytic at the origin and such that:

$$f(z) = h_j(\sqrt{1-8z}), \quad (z \to \frac{1}{8}^-),$$

By substitution $t := \sqrt{1-8z}$ we obtain the analytic at $t = 0$ function $h_j(t)$. Then $f_1 = h_j'(t = 0)$. □

Theorem 27. *Expansions of function f_K in neighborhoods of $z = \frac{1}{8}$ and $z = -\frac{1}{8}$ are as follows:*

$$f_K(z) = k_0^{(1)} + k_1^{(1)}\sqrt{1-8z} + ...$$

where

$$k_0^{(1)} = \frac{1}{8}\left(2 - \sqrt{2}\right), \quad k_1^{(1)} = -1$$

and

$$f_K(z) = k_0^{(2)} + k_1^{(2)}\sqrt{1+8z} + ...$$

where

$$k_0^{(2)} = \frac{1}{8}\left(2 - \sqrt{2}\right), \quad k_1^{(2)} = -1$$

Proof. Analogous to the proof of Theorem 26. □

Analogously we count the appropriate coefficients of generating function f_T of intuitionistic tautologies.

Theorem 28. *Expansions of function f_T in neighborhoods of $z = \frac{1}{8}$ and $z = -\frac{1}{8}$ are as follows:*

$$f_T(z) = t_0^{(1)} + t_1^{(1)}\sqrt{1-8z} + ...$$

where

$$t_0^{(\rho)} = 0.0517192..., \quad t_1^{(\rho)} = -0.4599704...$$

and

$$f_T(z) = t_0^{(2)} + t_1^{(2)}\sqrt{1+8z} + ...$$

where

$$t_0^{(-\rho)} = 0.0517192..., \quad t_1^{(-\rho)} = -0.4599704...$$

Proof. The above coefficients were found using *the Mathematica* package. □

Theorem 29. *The density of truth of equivalential reducts of classical and intuitionistic logic does not exist.*

Proof. Let us observe that the density of truth of the considered logics are given as the following limits (see Lemma 14):

$$\begin{aligned} \mu(CL_E) &= \lim_{n\to\infty} \frac{[z^n]\{f_K(z)\}}{[z^n]\{f(z)\}} = \lim_{n\to\infty} \frac{-1 - 1\cdot(-1)^n}{-4} \\ \mu(INT_E) &= \lim_{n\to\infty} \frac{[z^n]\{f_T(z)\}}{[z^n]\{f(z)\}} = \lim_{n\to\infty} \frac{-0.4599704... - 0.4599704...\cdot(-1)^n}{-4} \end{aligned}$$

It is easy to observe that the above limits do not exist. □

Let us examine the asymptotic behavior of the fractions of classical and intuitionistic tautologies (of even length) among all formulas of even length:

$$\lim_{n\to\infty} \frac{[z^{2n}]\{f_K(z)\}}{[z^{2n}]\{f(z)\}} = \lim_{n\to\infty} \frac{-\frac{1}{8} - \frac{1}{8}\cdot(-1)^{2n}}{-\frac{1}{2}} = \frac{1}{2} \tag{44}$$

$$\lim_{n\to\infty} \frac{[z^{2n}]\{f_T(z)\}}{[z^{2n}]\{f(z)\}} = \lim_{n\to\infty} \frac{-0.4599704... - 0.4599704...\cdot(-1)^{2n}}{-\frac{1}{2}} \approx 0.23. \tag{45}$$

(46)

□

From (44) and (46) we may conclude:

Corollary 30. *The probability of finding an intuitionistic equivalential tautology of even length among all equivalential formulas of such length is asymptotically about 23 %.*

Corollary 31. *The probability of finding a classical equivalential tautology of even length among all equivalential formulas of such length is asymptotically equal to 50 %.*

Corollary 32. *The relative probability of finding an intuitionistic equivalential tautology of even length among the classical ones is asymptotically about 46 %.*

The last result is quite surprising for us. Let us remaind that in the language with two variables but with implication the relative probability is about 97 % (see [Kostrzycka 2003]). Analogously, in the language with implication, negation and one variable the relative probability is about 93% (see [Kostrzycka and Zaionc 2004]). That means the connector of equivalence is much more distinguishable between intuitionistic and classical logics than the others well known operators.

References

[Flajolet and Odlyzko 1990] Flajolet, P. and Odlyzko, A. S.(1990) Singularity analysis of generating functions, *SIAM J. on Discrete Math.*, 3(2), 216-240.

[Flajolet and Sedgewick 2001] Flajolet, P. and Sedgewick, R. (2001) *Analitic combinatorics: functional equations, rational and algebraic functions*, INRIA, Number 4103, (2001).

[Kostrzycka 2003] Kostrzycka, Z. (2003) On the density of implicational parts of intuitionistic and classical logics, *Journal of Applied Non-Classical Logics*, Vol. 13, Number 3, 295-325.

[Kostrzycka and Zaionc 2004] Kostrzycka, Z. and Zaionc, M. (2004) Statistics of intuitionistic versus classical logics, *Studia Logica*, **76**, 307-328.

[Matecki 2005] Matecki, G. (2005) Asymptotic density for equivalence, *Electronic Notes in Theoretical Computer Science URL*, 140, 81-91.

[Moczurad, Tyszkiewicz and Zaionc 2000] Moczurad, M., Tyszkiewicz, J. and Zaionc, M. (2000) Statistical properties of simple types, *Mathematical Structures in Computer Science*, Vol. 10, 575-594.

[Robbins 1955] Robbins, H. A remark on Stirling formula, *Ammer. Math. Monthly*, 62, 26-29.

[Słomczyńska 1996] Słomczyńska, K. (1996) Equivalential algebras. Part I: Representation, *Algebra Universalis*, 35, 524-547.

[Szegö 1975] Szegö, G. (1975) Orthogonal polynomials. Fourth edition.*AMS, Colloquium Publications*, **23**, Providence.

[Wilf 1994] Wilf, H. S. (1994) *Generating functionology.* Second edition. Academic Press, Boston.

[Wroński 1993] Wroński, A. (1993) On the free equivalential algebras with three generators, *Bulletin of the Section of Logic*, 22, 37-39.

[Zaionc 2004] Zaionc, M. (2004) On the asymptotic density of autologies in logic of implication and negation, *Reports on Mathematical Logic*, **38**.