Statistics of intuitionistic versus classical logics *

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Abstract

For the given logical calculus we investigate the size of the proportion of the number of true formulas of a certain length n against the number of all formulas of such length. We are especially interested in asymptotic behavior of this fraction when n tends to infinity. If the limit of fractions exists it represents the real number between 0 and 1 which we may call *the density of truth* for the investigated logic. In this paper we apply this approach to the intuitionistic logic of one variable with implication and negation. The result is obtained by reducing it to the same problem of Dummett's intermediate linear logic of one variable (see [**?**]). Actually, this paper shows the exact density of intuitionistic logic and demonstrates that it covers a substantial part (more then 93%) of classical propositional calculus. Despite using strictly mathematical means to solve all discussed problems, this paper in fact, may have a philosophical impact on understanding how much the phenomenon of truth is sporadic or frequent in random mathematics sentences.

1 Introduction

The research described in this paper is a part of a project of quantitative investigations in logic. This paper is their continuation; especially methods of finding the asymptotic probability in some propositional logics are developed.

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For propositional formulas we investigate the size of the fraction of valid formulas of the given length n against the number of all formulas of that length. Our interest lays in finding a limit of that fraction when $n \to \infty$. If the limit exists it represents number which we may call *the density of truth* for the investigated logic . Probabilistic methods appear to be very powerful in logic, combinatorics and computer science. From a point of view of these methods it is enough to investigate a typical object chosen from a given set. In particular we are interested in finding the *"density"* of classes of formulas. We investigate the language $\mathcal{F}^{\{\to,\neg\}}$ consisting of implicational-negational formulas over one propositional variable. For some subclass of formulas $A \subset \mathcal{F}^{\{\to,\neg\}}$ we may associate the density $\mu(A)$ as:

$$\mu(A) = \lim_{n\to\infty} \frac{\#\{t \in A : \|t\| = n\}}{\#\{t \in \mathcal{F}^{\{\to,\neg\}} : \|t\| = n\}} \tag{1}$$

where $\|.\|$ stands for the length of formula defined in the conventional way as a number of characters. The number $\mu(A)$ if exists, is an asymptotic probability of finding formula from the class A among all formulas from $\mathcal{F}^{\{\to,\neg\}}$ and the asymptotic density of the set A in the set $\mathcal{F}^{\{\to,\neg\}}$ as well. The paper solves the problem of density for the set of all tautologies of Dummett's intermediate linear logic of one variable and equivalently for intuitionistic logic.

The paper is a natural continuation of the problem concerning the *density of truth* in classical logic of one variable. The result published in [**?**] proved the existence of the *density of truth* for classical (and intuitionistic) logic of implication of one variable. In the paper [**?**] it is shown that the *density* also exists for the implicational-negational classical formulas of one variable. In this paper we prove the similar result for Dummett's intermediate linear logic LC and intuitionistic logic.

2 Linear Dummett's logic

In this section we are going to study the quantitative relations in the implicational - negational fragment of Dummett logic. This particular fragment of Dummett's logic has been chosen for two simple reasons. First of all it is a good opportunity and possibility of interesting comparison with similar quantitative results already proved for the classical logic of the same language (see [**?**]). The second reason is that the Dummett logic reduced to implication and negation with one propositional variable is identical with the intuitionistic logic of this language. Therefore we can easily prove the result comparing intuitionistic and classical logics with respect of its quantitative and asymptotic behaviors.

Linear calculus LC was studied in [**?**] by Dummett and it seems to be a typical example of the intermediate propositional calculi. Syntactically the logic is obtained by adding the axiom $(p \to q) \vee (q \to p)$ to axioms of intuitionistic logic. The logic LC is define as a set of all consequences of new axioms by modus ponens and substitution rules. In [**?**] Dummett showed that this logic can be characterized by a denumerable matrix M_ω described bellow in definition **??**. The language of implicational - negational formulas of one propositional variable a consists of formulas $\mathcal{F}^{\{\to,\neg\}}$ built from a by means of negation and implication only.

$$\begin{array}{rcl} a & \in & \mathcal{F}^{\{\to,\neg\}} \\ \phi \to \psi \in \mathcal{F}^{\{\to,\neg\}} & iff & \phi \in \mathcal{F}^{\{\to,\neg\}} \text{ and } \psi \in \mathcal{F}^{\{\to,\neg\}} \\ \neg\phi \in \mathcal{F}^{\{\to,\neg\}} & iff & \phi \in \mathcal{F}^{\{\to,\neg\}}. \end{array}$$

Definition 1 *By Dummett's matrix we mean the infinite-valued characteristic matrix $M_\omega = \langle |M_\omega|, \sim, \Rightarrow, \{1\}\rangle$, where the set $|M_\omega| = \mathbb{N} \cup \{\omega\}$ is equipped with two operations $\{\sim, \Rightarrow\}$ defined as:*

$$\sim p = \left\{ \begin{array}{lll} \omega & gdy & p < \omega \\ 1 & gdy & p = \omega \end{array} \right. \qquad p \Rightarrow q = \left\{ \begin{array}{lll} 1 & gdy & p \geq q \\ q & gdy & p < q. \end{array} \right.$$

Definition 2 *By the valuation of our language $\mathcal{F}^{\{\to,\neg\}}$ in the matrix $|M_\omega|$ we mean any function $v : \mathcal{F}^{\{\to,\neg\}} \to |M_\omega|$ satisfying $v(\phi \to \psi) = v(\phi) \Rightarrow v(\psi)$ and $v(\neg\phi) =\sim v(\phi)$. A formula α is a tautology iff $v(\alpha) = 1$ for every valuation $v : \mathcal{F}^{\{\to,\neg\}} \to |M_\omega|$. By $E(M_\omega)$ we mean the set of all tautologies in LC. Since we have formulas built with exactly one propositional variable a we can enumerate valuations by the elements of $|M_\omega|$ as follows*

$$v_i(a) \quad = \quad i \quad \text{for all } i \in |M_\omega|. \tag{2}$$

Definition 3 *By the sequence of valuations α we mean any function $\alpha : |M_\omega| \to |M_\omega|$. Sequences of valuations are ordered componentwise by $\alpha \leq \beta$ iff for all $i \in |M_\omega| \quad \alpha(i) \leq \beta(i)$ and form a poset. On sequences we may introduce operations $\{\sim, \Rightarrow\}$ also componentwise allowing that : $\alpha \Rightarrow \beta$ is a new sequence of valuations such that $(\alpha \Rightarrow \beta)(i) = \alpha(i) \Rightarrow \beta(i)$ and $\sim \alpha$ is a sequence defined $(\sim \alpha)(i) =\sim \alpha(i)$.*

We partition the set of all formulas into several classes according to formulas behavior on all possible evaluations.

Definition 4 *Each sequence of valuations α defines uniquely the set of formulas $F^\alpha \subset \mathcal{F}^{\{\to,\neg\}}$ which are undistinguishable by all valuations.*

$$F^\alpha \quad = \quad \left\{ \phi \in \mathcal{F}^{\{\to,\neg\}} : \forall i \in |M_\omega| \quad v_i(\phi) = \alpha(i) \right\}. \tag{3}$$

For example, the initial formula a belongs to the class F^α for the sequence $\alpha(i) = i, \quad \forall i \in |M_\omega|$, while the formula $\neg a \to a$ lays in the class F^α for the sequence $\alpha(i) = 1, \quad \forall i \in \mathbb{N}$ and $\alpha(\omega) = \omega$. It is obvious that classes are disjoint so $F^\alpha \cap F^\beta = \emptyset$ for $\alpha \neq= \beta$ and $\bigcup_\alpha F^\alpha = \mathcal{F}^{\{\to,\neg\}}$. Some sequences has no realization among formulas or there is no formula which behaves the way described by sequence.

Definition 5 *The sequence α is called nonempty if the set of formulas $F^\alpha \neq \emptyset$.*

Our first task is to separate all nonempty sequences of valuations. We can easily see that the class F^α for the sequence $\alpha(i) = i, \quad \forall i \in |M_\omega|$ is nonempty since our initial formula a lays in F^α.

Definition 6 *Closing the set of nonempty sequences of valuations by operations $\{\sim, \Rightarrow\}$ we isolate exactly six sequences. Bellow we make a list of all six classes together with appropriate sequences. In order to simplify notations we are going to call classes A, B, C, D, E, G:*

$$\begin{array}{lcll} A & = & F^{\alpha_A} & \alpha_A(i) = \omega, \\ B & = & F^{\alpha_B} & \alpha_B(i) = i, \\ C & = & F^{\alpha_C} & \alpha_C(i) = \omega \textit{ for } = i < \omega \textit{ and } \alpha_C(\omega) = 1, \\ D & = & F^{\alpha_D} & \alpha_D(i) = i \textit{ for } i < \omega \textit{ and } \alpha_D(\omega) = 1, \\ E & = & F^{\alpha_E} & \alpha_E(i) = 1 \textit{ for } i < \omega \textit{ and } \alpha_E(\omega) = \omega, \\ G & = & F^{\alpha_G} & \alpha_G(i) = 1. \end{array}$$

As we can see the class G establishes the set $E(M_\omega)$ of all tautologies in LC.

Lemma 7 *(Dummett [**?**]) Let $\equiv$ be an equivalence relation over $\mathcal{F}^{\{\to,\neg\}}$ such that: $\phi \equiv \psi$ iff $\phi \to \psi$ and $\psi \to \phi$ are provable in the linear logic LC . The relation $\equiv$ is a congruence relation on $\mathcal{F}^{\{\to,\neg\}}$. The quotient algebra $\mathcal{F}^{\{\to,\neg\}}/_\equiv$ consists of following six congruence classes:*

$$\begin{array}{lcl} A & = & [\neg(p \to p)]_\equiv, \\ B & = & [p]_\equiv, \\ C & = & [\neg p]_\equiv, \\ D & = & [(\neg p \to p) \to p]_\equiv, \\ E & = & [\neg p \to p]_\equiv, \\ G & = & [p \to p]_\equiv. \end{array}$$

Proof. The proof is a trivial consequence of the Dummett completeness theorem (see [**?**]). Representatives of congruence classes chosen in the table above are the shortest with respect of length of formula which is defined in the definition **??** in the section **??**. □

Definition 8 *Semantic operations $\{\sim, \Rightarrow\}$ on these classes defined by $F^\alpha \Rightarrow F^\beta = F^{\alpha \Rightarrow \beta}$ and $\sim F^\alpha = F^{\sim\alpha}$ can be displayed by the following truth table:*

$\Rightarrow$	A	B	C	D	E	G	$\sim$
A	G	G	G	G	G	G	G
B	C	G	C	G	G	G	C
C	E	E	G	G	E	G	E
D	A	E	C	G	E	G	A
E	C	D	C	D	G	G	C
G	A	B	C	D	E	G	A

Table 1.

Definition 9 *The order on classes F^{α} is defined as $F^{\alpha} \leq F^{\alpha'}$ iff $\alpha \geq \alpha'$. It forms the following lattice diagram with the class of tautologies G being on the top:*

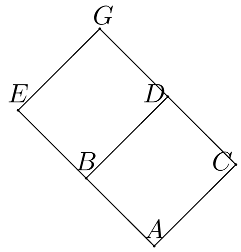

Diagram 1.

For technical reasons we are also going to consider two posets obtained from the one above by appropriate identification.

Definition 10 *Let us define three elements chain obtained from the poset above by the identification of classes E and G, B and D as well as classes A and C. We will name such classes as $EG = E \cup G$, $BD = B \cup D$ and $AC = A \cup C$. Let us define also four elements Boolean algebra, obtained from the poset above by identifying classes D and G, and B and E. Accordingly we will call such classes $DG = D \cup G$ and $BE = B \cup E$. They have following diagrams:*

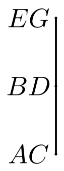

Diagram 2

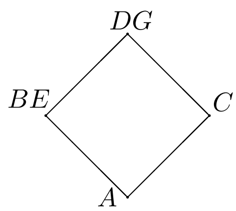

Diagram 3

Observation 11 *The operations $\{\sim, \Rightarrow\}$ on new classes in new posets are given by the following truth tables:*

$\Rightarrow$	AC	BD	EG	$\sim$
AC	EG	EG	EG	EG
BD	AC	EG	EG	AC
EG	AC	BD	EG	AC

Table 2.

$\Rightarrow$	A	C	BE	DG	$\sim$
A	DG	DG	DG	DG	DG
C	BE	DG	BE	DG	BE
BE	C	C	DG	DG	C
DG	A	C	BE	DG	A

Table 3.

As we can observe the first truth table describes operations in Gödel 3 valued matrix, while the second one is a matrix of all valuations associated with the standard classical logic of one variable.

Lemma 12 *The matrix described in Table 3 is a matrix for the classical propositional logic of implication and negation with one variable.*

Proof. We start the proof by splitting the set of all formulas into four classes according to behavior of formulas on two possible evaluations. Since we have formulas built with exactly one propositional variable a we can evaluate formulas by two valuations: ν_0 associating 0 to a and ν_1 associating 1 to a. For any $i, j \in \{0,1\}$ by $F^{i,j}$ we mean the set of formulas ϕ from $\mathcal{F}^{\{\rightarrow,\neg\}}$ such that $\nu_0(\phi) = i$ and $\nu_1(\phi) = j$. The four classes $F^{i,j}$ are ordered by assuming that $F^{i,j} \leq F^{i',j'}$ iff $i \leq i'$ and $j \leq j'$. On our four classes $F^{i,j}$ we can establish operation of implication $\Rightarrow$ by $F^{i,j} \Rightarrow F^{i',j'} = F^{i\rightarrow i', j\rightarrow j'}$ where $\rightarrow$ stands for the classical implication defined on the set $\{0,1\}$. In the same way we define the operation of negation by $\sim F^{i,j} = F^{\neg i, \neg j}$. By the construction above we get the four element Boolean algebra, exactly the same as we can see in the diagram 3 with the identical truth table as described in Table 3. The classes are following:

$$\begin{aligned} A &= F^{0,0}, \\ BE &= F^{0,1}, \\ C &= F^{1,0}, \\ DG &= F^{1,1}. \end{aligned}$$

and the truth table is identical with the one defined by operations $\{\Rightarrow, \sim\}$ described above. Therefore the results concerning asymptotic probabilities associated with any of four classes are identical with those for classical propositional logic. This result has been studied in [**?**]. Especially theorem 5.4 in [**?**] shows what exactly is the asymptotic density of the class DG. □

3 Counting Formulas

In this section we present some properties of numbers characterizing the amount of formulas in different classes defined in our language. First, let us establish the way of measuring the length of formulas.

Definition 13 *By $\|\phi\|$ we mean the length of the formula ϕ, which is the total number of characters in the formula, including implication and negation signs. Parentheses, which are sometimes necessary, are not included in the length of formula. Formally*

$$\begin{aligned} \|a\| &= 1, \\ \|\phi \rightarrow \psi\| &= \|\phi\| + \|\psi\| + 1, \\ \|\neg\phi\| &= \|\phi\| + 1. \end{aligned}$$

Definition 14 *By $\mathcal{F}_n^{\{\rightarrow,\neg\}}$ we mean the set of formulas of length $n-1$. Subclasses $A_n, B_n, C_n, D_n, E_n, G_n$ and additional subclasses EG_n, BD_n, AC_n, DG_n, BE_n of formulas of length $n-1$ are defined accordingly by:*

$$
\begin{array}{rclcrcl}
A_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap A, & & B_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap B, \\
C_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap C, & & D_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap D, \\
E_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap E, & & G_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap G, \\
EG_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap EG, & & BD_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap BD, \\
AC_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap AC, & & DG_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap DG, \\
BE_n & = & \mathcal{F}_n^{\{\to,\neg\}} \cap BE. & & & &
\end{array}
$$

We can see that for any $n \in \mathbb{N}$ the number of formulas in $\mathcal{F}_n^{\{\to,\neg\}}$ is finite and will be denoted as $\left|\mathcal{F}_n^{\{\to,\neg\}}\right|$. Consequently all subclasses listed above are also finite for all y $n \in \mathbb{N}$.

4 Generating functions

The main tool we use for dealing with asymptotics of sequences of numbers are *generating functions.* A nice exposition of the method can be found in [**?**] and [**?**]. Our main task in this paper is to determine limits of various sequences of real numbers. For this purpose combinatorics has developed an extremely powerful tool, in the form of generating series and generating functions. Let $A = (A_0, A_1, A_2, \ldots)$ be a sequence of real numbers. The *ordinary generating series* for A is the formal power series $\sum_{n=0}^{\infty} A_n z^n$. And, of course, formal power series are in one-to-one correspondence to sequences. However, considering z as a complex variable, this series, as it is known from the theory of analytic functions, converges uniformly to a function $f_A(z)$ in some open disc $\{z \in \mathcal{C} : |z| < R\}$ of maximal diameter, and $R \geq 0$ is called its radius of convergence. So with the sequence A we can associate a complex function $f_A(z)$, called the *ordinary generating function* for A, defined in a neighborhood of 0. This correspondence is one-to-one again (unless $R = 0$), since the expansion of a complex function $f(z)$, analytic in a neighborhood of z_0, into a power series $\sum_{n=0}^{\infty} A_n (z - z_0)^n$ is unique, and moreover, this series is the Taylor series, given by

$$
A_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0). \tag{4}
$$

Many questions concerning the asymptotic behavior of A can be efficiently resolved by analyzing the behavior of f_A at the complex circle $|z| = R$.
This is the approach we take to determine the asymptotic fraction of tautologies and many other classes of formulas among all formulas of a given length.
The key tool will be the following result due to Szegö [**?**] [Thm. 8.4], see as well [**?**] [Thm. 5.3.2] which relates the generating functions of numerical sequences with limit of the fractions investigated. For the technique of proof described below please consult also [**?**] as well as [**?**]. We need the following much simpler version of the Szegö Lemma.

Lemma 15 *Let $v(z)$ be analytic in $|z| < 1$ with $z = 1$ the only singularity at the circle $|z| = 1$. If $v(z)$ in the vicinity of $z = 1$ has an expansion of the form*

$$v(z) = \sum_{p \geq 0} v_p (1-z)^{\frac{p}{2}}, \tag{5}$$

where $p > 0$, and the branch chosen above for the expansion equals to $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = v_1 \binom{1/2}{n} (-1)^n + O(n^{-2}). \tag{6}$$

The symbol $[z^n]\{v(z)\}$ stands for the coefficient of z^n in the exponential series expansion of $v(z)$.

For technical reasons we will need to know the rate of grow of the function $\binom{1/2}{n}(-1)^n$ which appears at the formula (**??**)

Lemma 16 $\binom{1/2}{n}(-1)^{n+1} = O(n^{-3/2})$

Proof. . It can be obtained by the Stirling approximation formula (see [**?**] for details, consult also lemma 7.5 page 589 at [**?**]).

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}. \tag{7}$$

□

5 Calculation of limits

In this section we are going to find the method of finding asymptotic densities for the classes of formulas for which the generating functions are already calculated. The main tool used for this purpose is theorem based on simplified Szegö lemma. The following theorem is a main tool for finding limits of the fraction $\frac{a_n}{b_n}$ when generating functions for sequences a_n and b_n satisfies conditions of simplified Szegö lemma **??**.

Lemma 17 *Suppose two functions $v(z)$ and $w(z)$ satisfies assumptions of simplified Szegö theorem (lemma **??**) i.e. both v and w are analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$. Both $v(z)$ and $w(z)$ in the vicinity of $z = 1$ have expansions of the form*

$$v(z) = \sum_{p \geq 0} v_p (1-z)^{p/2}, \tag{8}$$

$$w(z) = \sum_{p \geq 0} w_p (1-z)^{p/2}, \tag{9}$$

then the limit of $\frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}}$ *exists and is given by formula:*

$$\lim_{n\to\infty} \frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}} = \frac{v_1}{w_1} \tag{10}$$

Proof. Applying the main formula from simplified version of Szegö lemma **??** and equation from lemma **??**

$$\begin{aligned}
\lim_{n\to\infty} \frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}} &= \lim_{n\to\infty} \frac{v_1 \binom{1/2}{n}(-1)^n + O(n^{-2})}{w_1 \binom{1/2}{n}(-1)^n + O(n^{-2})} \\
&= \lim_{n\to\infty} \frac{v_1 O(n^{-3/2}) + O(n^{-2})}{w_1 O(n^{-3/2}) + O(n^{-2})} = \frac{v_1}{w_1}
\end{aligned}$$

□

Theorem 18 *Suppose two functions* $v(z)$ *and* $w(z)$ *satisfies assumptions of simplified Szegö theorem (lemma* **??***) i.e. both* v *and* w *are analytic in* $|z| < 1$ *with* $z = 1$ *being the only singularity at the circle* $|z| = 1$*. Both* $v(z)$ *and* $w(z)$ *in the vicinity of* $z = 1$ *have expansions of the form*

$$v(z) = \sum_{p\geq 0} v_p (1-z)^{p/2}, \tag{11}$$

$$w(z) = \sum_{p\geq 0} w_p (1-z)^{p/2}, \tag{12}$$

Suppose we have functions $\widetilde{v}$ *and* $\widetilde{w}$ *satisfying* $\widetilde{v}(\sqrt{1-z}) = v(z)$ *and* $\widetilde{w}(\sqrt{1-z}) = w(z)$ *then the limit of* $\frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}}$ *exists and is given by formula:*

$$\lim_{n\to\infty} \frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}} = \frac{(\widetilde{v})'(0)}{(\widetilde{w})'(0)} \tag{13}$$

Proof. Simple consequence of lemma **??**. New functions $\widetilde{v}$ and $\widetilde{w}$ have expansions

$$\widetilde{v}(z) = \sum_{p\geq 0} v_p z^p, \tag{14}$$

$$\widetilde{w}(z) = \sum_{p\geq 0} w_p z^p, \tag{15}$$

Therefore $v_1 = (\widetilde{v})'(0)$ and $w_1 = (\widetilde{w})'(0)$. By lemma **??** the result (**??**) is obvious.

□

6 Calculating generating functions

The main goal of this section is to find the generating function for the class of tautologies G meaning that we build the function for the sequence of numbers $|G_n|$ and contrast it with the generating function of the class of all formulas $\left|\mathcal{F}_n^{\{\rightarrow,\neg\}}\right|$. First, recall the following two generating functions calculated in [?] for sequences $\left|\mathcal{F}_n^{\{\rightarrow,\neg\}}\right|$ and $|DG_n|$. We start by calculating generating functions we will need.

Definition 19 *The number F_n is given by the recursion:*

$$F_0 = 0, \quad F_1 = 0, \quad F_2 = 1, \tag{16}$$

$$F_n = F_{n-1} + \sum_{i=1}^{n-1} F_i F_{n-i}. \tag{17}$$

Lemma 20 *The number of formulas of length $n-1$ is F_n. So $F_n = \left|\mathcal{F}_n^{\{\rightarrow,\neg\}}\right|$.*

Proof. Any formula of length $n-1$ for $n > 2$ is either a negation of some formula of length $n-2$ for which responsible is the fragment F_{n-1}, or is the implication between some pair of formulas of lengths $i-1$ and $n-i-1$, respectively. The length of any of such implicational formulas must be $(i-1)+(n-i-1)+1$ which is exactly $n-1$. Therefore the total number of such formulas is $\sum_{i=1}^{n-1} F_i F_{n-i}$. □

Lemma 21 *The generating function f_F for the numbers F_n is*

$$f_F(z) = \frac{1-z}{2} - \frac{\sqrt{(z+1)(1-3z)}}{2}. \tag{18}$$

Proof. The recurrence $F_n = F_{n-1} + \sum_{i=1}^{n-2} F_i F_{n-i}$ becomes the equality

$$f_F(z) = z f_F(z) + f_F^2(z) + z^2 \tag{19}$$

since the recursion fragment $\sum_{i=1}^{n-2} F_i F_{n-i}$ exactly corresponds to the multiplication of power series. The term F_{n-1} corresponds to the function $z f_F(z)$. The quadratic term z^2 corresponds to the first non-zero coefficient in the power series of f_F. Solving the equation we get two possible solutions: $f_F(z) = (1-z)/2 - \sqrt{-3z^2 - 2z + 1}/2$ or $f_F(z) = (1-z)/2 + \sqrt{-3z^2 - 2z + 1}/2$. We have to choose the first solution, since it corresponds to the assumption $f_F(0) = 0$ (see equation (??)). □

Lemma 22 *(Zaionc [?]) The generating function f_{DG} for the sequence of numbers $|DG_n|$ is:*

$$\begin{aligned} f_{DG}(z) &= \frac{1}{24}\left(24-\sqrt{2}Z-\sqrt{2}T-2\sqrt{9-90z+27z^2+Y+ZT}\right), \qquad (20)\\ \textit{where} & \\ X &= \sqrt{(3z+3)(1-3z)},\\ Y &= \sqrt{3}(3z-3)X,\\ Z &= \sqrt{9+54z-9z^2+Y},\\ T &= \sqrt{9+54z+63z^2+Y}. \end{aligned}$$

Proof. Calculation of the function together with the asymptotic density of the class of classical implicational-negational tautologies can be found in [?].

The following part of this chapter is dedicated to the problem of construction of generating functions for classes from the Gödel 3 valued matrix. Let us start with the proof of correctness of the mutual recurrence relations between three classes $|AC_n|$, $|BD_n|$ and $|EG_n|$.

Lemma 23 *The numbers $|AC_n|$, $|BD_n|$ and $|EG_n|$ are given by the following mutual recursions:*

$$\begin{aligned} |AC_0| &= |AC_1| = |AC_2| = 0, \quad |AC_3| = 1,\\ |AC_n| &= |BD_{n-1}| + |EG_{n-1}| + \sum_{i=1}^{n-2}(|BD_i| + |EG_i|)|AC_{n-i}|, \qquad (21)\\ |BD_0| &= |BD_1| = 0, \quad |BD_2| = 1,\\ |BD_n| &= \sum_{i=1}^{n-2}|EG_i||BD_{n-i}|, \qquad (22)\\ |EG_0| &= |EG_1| = |EG_2| = |EG_3| = 0, \ |EG_4| = 2,\\ |EG_n| &= |F_n| - (|AC_n| + |BD_n|). \qquad (23) \end{aligned}$$

Proof. It follows easily from Table 2. Formulas from class AC described in by formula (**??**), can be obtain as negations of these from classes BD and EG. This part is responsible for the component $|BD_{n-1}|+|EG_{n-1}|$. Analyzing table 2 we also can notice that the number of formulas from AC in the form of implications depends only on the same classes BD and EG and these from AC. This fact is described in the fragment $\sum_{i=1}^{n-2}(|BD_i| + |EG_i|)|AC_{n-i}|$. So, we have proved (**??**). The formulas in the class BD can only be implications of formulas from classes EG and BD. This gives (**??**). The last equation (**??**) is obvious. □

Lemma 24 *The generating function f_{BD} for sequence of numbers $|BD_n|$ is:*

$$f_{BD}(z) = \frac{1}{24}\Big(3z - 3 - \sqrt{3}\,X - \sqrt{2}\,U + \sqrt{12\,(3+6z+45z^2) - 2\sqrt{2}\,(3z-3)\,U - 4Y + 2\sqrt{6}XU}\Big), \quad (24)$$

where

$$X = \sqrt{(3z+3)(1-3z)},$$
$$Y = \sqrt{3}(3z-3)X,$$
$$U = \sqrt{9+54z-9z^2-Y}.$$

Proof. First, we can observe that the generating functions f_{AC}, f_{BD} i f_{EG} for sequences $|AC_n|$, $|BD_n|$ and $|EG_n|$ satisfies the following equalities

$$f_{AC} = (f_{BD}+f_{EG})z + (f_{BD}+f_{EG})f_{AC}, \quad (25)$$
$$f_{BD} = f_{EG}\cdot f_{BD} + z^2, \quad (26)$$
$$f_{EG} = f_F - (f_{AC}+f_{BD}). \quad (27)$$

The recurrence (**??**) corresponds to multiplication of power series and then gives the equality (**??**). The quadratic term z^2 in (**??**) corresponds to the first non-zero coefficient in the power series of f_{BD}. In the same manner we can see that the fragment $\sum_{i=1}^{n-2}(|BD_i|+|EG_i|)|AC_{n-i}|$ corresponds to the multiplication $(f_{BD}+f_{EG})f_{AC}$, while the term $|BD_{n-1}|+|EG_{n-1}|$ corresponds to the function $(f_{BD}+f_{EG})z$. Solving this system of multi quadratic equations (**??**), (**??**) and (**??**) we obtain four solutions for f_{BD}. Note, that we choose the solution satisfying the boundary conditions $f_{BD}(0)=0$ and $f'_{BD}(0)=0$ which is presented in (**??**). □

Now, we are ready to attack the problem of finding the generating function for the class of tautologies G. The simplest way is to choose first the class B since it occurs only once in the table 1 as a result of implication between classes G and B again.

Lemma 25 *The generating function for the sequence $|B_n|$ is*

$$f_B = \frac{1}{2}(1 - f_{DG} + f_{BD} - \sqrt{(f_{DG}-f_{BD}-1)^2 - 4z^2}). \quad (28)$$

Proof. We start the proof with analyzing the main truth table 1. This suggests the recursion schema for this class which must be:

$$|B_0| = |B_1| = 0, \quad |B_2| = 1,$$
$$|B_n| = \sum_{i=1}^{n-2} |G_i||B_{n-i}|. \quad (29)$$

This can be translated into equation:

$$f_B = f_G f_B + z^2. \tag{30}$$

Just from the disjointness of classes we get

$$f_G + f_D = f_{DG}, \tag{31}$$
$$f_D + f_B = f_{BD}. \tag{32}$$

Therefore the generating function for the class of tautologies G can be presented in terms of already known generating functions f_{DG} and f_{BD} as follows:

$$f_G = f_{DG} - f_{BD} + f_B. \tag{33}$$

Therefore from (**??**) and (**??**) we get a quadratic equation with unknown function f_B, namely:

$$f_B = (f_{DG} - f_{BD} + f_B)f_B + z^2. \tag{34}$$

By solving (**??**) with the boundary condition $f_B(0) = 0$ we get function f_B which is presented here in terms of already known functions f_{DG} and f_{BD} (see (**??**) and (**??**)) . □

Lemma 26 *Generating function for the sequence of tautologies $|G_n|$ is:*

$$f_G = \frac{1}{2}(f_{DG} - f_{BD} + 1 - \sqrt{(f_{DG} - f_{BD} - 1)^2 - 4z^2}). \tag{35}$$

Proof. Trivially follows from (**??**) and (**??**). For simplicity, we can present it again in terms of functions already known: f_{DG} and f_{BD}. □

7 From generating functions to asymptotic densities

Definition 27 *In order to apply simplified Szegö lemma we have to have functions which are analytic in the open disc $|z| < 1$, and the nearest singularity is at $z_0 = 1$. For that purpose we are going to calibrate functions f_F and f_G in the following way*

$$\widehat{f_F}(z) = f_F\left(\tfrac{z}{3}\right), \qquad \widehat{f_{DG}}(z) = f_{DG}\left(\tfrac{z}{3}\right),$$
$$\widehat{f_{BD}}(z) = f_{BD}\left(\tfrac{z}{3}\right), \qquad \widehat{f_G}(z) = f_G\left(\tfrac{z}{3}\right).$$

After appropriate simplification of that expressions we get

$$
\begin{aligned}
\widehat{X} &= \sqrt{(z+3)(1-z)},\\
\widehat{Y} &= \sqrt{3}(z-3)\widehat{X},\\
\widehat{Z} &= \sqrt{9+18z-z^2+\widehat{Y}},\\
\widehat{T} &= \sqrt{9+18z+7z^2+\widehat{Y}},\\
\widehat{U} &= \sqrt{9+18z-z^2-\widehat{Y}},\\
\widehat{f_F}(z) &= \frac{1}{6}\left(3-z-\sqrt{3}\sqrt{(z+3)(1-z)}\right), && (36)\\
\widehat{f_{DG}}(z) &= \frac{1}{24}\left(24-\sqrt{2}\widehat{Z}-\sqrt{2}\widehat{T}-2\sqrt{9-30z+3z^2+\widehat{Y}+\widehat{Z}\widehat{T}}\right), && (37)\\
\widehat{f_{BD}}(z) &= \frac{1}{24}\Big(z-3-\sqrt{3}\,\widehat{X}-\sqrt{2}\,\widehat{U}+ && (38)\\
&\quad \sqrt{12\,(3+2z+5z^2)-2\sqrt{2}\;(z-3)\,\widehat{U}-4\widehat{Y}+2\sqrt{6}\widehat{X}\widehat{U}}\Big), && (39)\\
\widehat{f_G}(z) &= \frac{1}{2}(\widehat{f_{DG}}(z)-\widehat{f_{BD}}(z)+1-\\
&\quad \sqrt{(\widehat{f_{DG}}(z)-\widehat{f_{BD}}(z)-1)^2-\frac{4}{9}z^2}).
\end{aligned}
$$

Note that the relations between power series of appropriate functions are such as $[z^n]\{f(z)\} = \left([z^n]\{\widehat{f}(z)\}\right)3^n$.

Lemma 28 $z_0 = 1$ *is the only singularity of* $\widehat{f_F}$ *and* $\widehat{f_G}$ *located in* $|z| \leq 1$.

Proof. It is easy to observe the function $\widehat{f_F}(z)$ has the only singularities at $z = 1$ and $z = -3$. To make sure the function $\widehat{f_G}(z)$ has the nearest one in $z = 1$, we had to solve the following complicated equations:

$$
\begin{aligned}
9+18z-z^2+\widehat{Y} &= 0 && (40)\\
9+18z+7z^2+\widehat{Y} &= 0 && (41)\\
9+18z-z^2-\widehat{Y} &= 0 && (42)\\
9-30z+z^2+\widehat{Y}+\sqrt{9+18z-z^2+\widehat{Y}}\sqrt{9+18z+7z^2+\widehat{Y}} &= 0 && (43)\\
12\left(3+2z+5z^2\right)-2\sqrt{2}\;(z-3)\,\widehat{U}-4\widehat{Y}+2\sqrt{6}\widehat{X}\widehat{U} &= 0 && (44)\\
(\widehat{f_{DG}}(z)-\widehat{f_{BD}}(z)-1)^2-\frac{4}{9}z^2 &= 0 && (45)
\end{aligned}
$$

where $\widehat{X}, \widehat{Y}, \widehat{U}, \widehat{f_{BD}}$ and $\widehat{f_{DG}}$ are functions defined above in the definition (**??**) To do that we had extensively used *Mathematica* package and it occurred that all solutions which are different from $z = 1$ are situated outside the disc $|z| \leq 1$. □

Since functions $\widehat{f_F}$ and $\widehat{f_G}$ satisfies assumptions of simplified Szegö lemma (??) i.e. both $\widehat{f_F}$ and $\widehat{f_G}$ are analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$ we can utilize theorem ??. Therefore we need to find functions $\widetilde{f_F}$ and $\widetilde{f_G}$ satisfying $\widetilde{f_F}(\sqrt{1-z}) = \widehat{f_F}(z)$ and $\widetilde{f_G}(\sqrt{1-z}) = \widehat{f_G}(z)$. Additionally we will need also functions $\widetilde{f_{DG}}$, $\widetilde{f_{BD}}$ as well as supplementary functions $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{U}$, $\widetilde{Z}$ and $\widetilde{T}$. Functions $\widetilde{X}$, $\widetilde{Y}$,$\widetilde{U}$, $\widetilde{Z}$ and $\widetilde{T}$ are not having any specific combinatorial interpretation but are used only for the purpose of simplifying expressions for more complicated functions including $\widetilde{f_G}$. Let we define tilded functions as follows:

Definition 29

$$
\begin{aligned}
\widetilde{X}(z) &= z\sqrt{4-z^2}, && (46)\\
\widetilde{Y}(z) &= -\sqrt{3}(z^2+2)\widetilde{X}(z), && (47)\\
\widetilde{Z}(z) &= \sqrt{26-16z^2-z^4+\widetilde{Y}(z)}, && (48)\\
\widetilde{T}(z) &= \sqrt{34-32z^2+7z^4+\widetilde{Y}(z)}, && (49)\\
\widetilde{U}(z) &= \sqrt{26-16z^2-z^4-\widetilde{Y}(z)}, && (50)\\
\widetilde{f_F}(z) &= \frac{1}{6}\left(2+z^2-\sqrt{3}z\sqrt{4-z^2}\right), && (51)\\
\widetilde{f_{DG}}(z) &= \frac{1}{24}\left(24-\sqrt{2}\widetilde{Z}-\sqrt{2}\widetilde{T}-2\sqrt{-18+24z^2+3z^4+\widetilde{Y}+\widetilde{Z}\widetilde{T}}\right), && (52)\\
\widetilde{f_{BD}}(z) &= \frac{1}{24}\left(-z^2-2-\sqrt{3}\widetilde{X}-\sqrt{2}\widetilde{U}+\right. && (53)\\
&\qquad \left.\sqrt{2}\sqrt{6(10-12z^2+5z^4)-2\widetilde{Y}+\sqrt{2}\left(2+z^2+\sqrt{3}\widetilde{X}\right)\widetilde{U}}\right), \\
\widetilde{f_G}(z) &= \frac{1}{2}(\widetilde{f_{DG}}(z)-\widetilde{f_{BD}}(z)+1- \\
&\qquad \sqrt{(\widetilde{f_{DG}}(z)-\widetilde{f_{BD}}(z)-1)^2-\frac{4}{9}(1-z^2)^2}). && (54)
\end{aligned}
$$

Lemma 30 *New tilded functions introduced in definition* ?? *are such that:*

$$
\begin{aligned}
\widetilde{f_{DG}}(\sqrt{1-z}) &= \widehat{f_{DG}}(z),\\
\widetilde{f_{BD}}(\sqrt{1-z}) &= \widehat{f_{BD}}(z),\\
\widetilde{f_F}(\sqrt{1-z}) &= \widehat{f_F}(z),\\
\widetilde{f_G}(\sqrt{1-z}) &= \widehat{f_G}(z)
\end{aligned}
$$

Proof. Trivially follows from the definition of functions. It can be seen if we replace expression $\sqrt{1-z}$ everywhere in functions in the definition ?? by the new fresh variable t. And at the same time replace z by $1-t^2$.

Theorem 31 *[Main limit theorem] The density the class of tautologies in the Dummett's logic exists and is given by the following expression:*

$$\mu(G) \quad = \quad \lim_{n\to\infty} \frac{|G_n|}{|\mathcal{F}_n^{\{\to,\neg\}}|} = \frac{(\widetilde{f_G})'(0)}{(\widetilde{f_F})'(0)} \approx 0.395205 \ .$$

Proof. Suppose g_1 and f_1 are first terms of the expansions of functions $\widehat{f_g}(z) = \sum_{p\geq 0} g_p(1-z)^{p/2}$ and $\widehat{f_F}(z) = \sum_{p\geq 0} f_p(1-z)^{p/2}$ in the vicinity of $z=1$, the only singularity of both functions at the circle $\|z\|=1$. Putting together lemma **??**, theorem **??** and results in lemma **??** we get:

$$\begin{aligned} \lim_{n\to\infty} \frac{|G_n|}{|\mathcal{F}_n^{\{\to,\neg\}}|} &= \lim_{n\to\infty} \frac{(g_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n} \\ &= \lim_{n\to\infty} \frac{g_1}{f_1}(1+o(1)) = \frac{g_1}{f_1} = \frac{(\widetilde{f_G})'(0)}{(\widetilde{f_F})'(0)} \ . \end{aligned}$$

The derivatives $(\widetilde{f_G})'(0)$ and $(\widetilde{f_F})'(0)$ have been found analytically using *Mathematica* package. Numerically the value $\frac{(\widetilde{f_G})'(0)}{(\widetilde{f_F})'(0)}$ is about 0.395205... The exact value of $\frac{(\widetilde{f_G})'(0)}{(\widetilde{f_F})'(0)}$ is:

$$\frac{1}{2\sqrt{3}}\left(\widetilde{f'_{BD}}(0) - \widetilde{f'_{DG}}(0) + \frac{\left(1+\widetilde{f_{BD}}(0)-\widetilde{f_{DG}}(0)\right)\left(\widetilde{f'_{BD}}(0)-\widetilde{f'_{DG}}(0)\right)}{\sqrt{\left(\widetilde{f_{DG}}(0)-\widetilde{f_{BD}}(0)-1\right)^2-\frac{4}{9}}}\right), \tag{55}$$

where values of functions at 0 and derivatives at 0 are computed separately and

$$\widetilde{f_{BD}}(0) = \frac{1}{24}\left(\sqrt{120+8\sqrt{13}} - 2\sqrt{13} - 2\right), \tag{56}$$

$$\widetilde{f'_{BD}}(0) = \frac{7}{2\sqrt{78\left(15+\sqrt{13}\right)}} + \frac{1}{\sqrt{360+24\sqrt{13}}} - \frac{13+\sqrt{13}}{52\sqrt{3}}, \tag{57}$$

$$\widetilde{f_{DG}}(0) = \frac{1}{24}\left(24 - 2\sqrt{13} - 2\sqrt{17} - 2\sqrt{2\sqrt{221}-18}\right), \tag{58}$$

$$\widetilde{f'_{DG}}(0) = -\frac{1}{2}\sqrt{\frac{1593+107\sqrt{221}+\sqrt{70\left(39737+2673\sqrt{221}\right)}}{23205}}. \tag{59}$$

The density distribution for the Dummett logic is displayed on the following diagram. Calculation of this density for other classes have been done in the very same way as for the class of tautologies G.

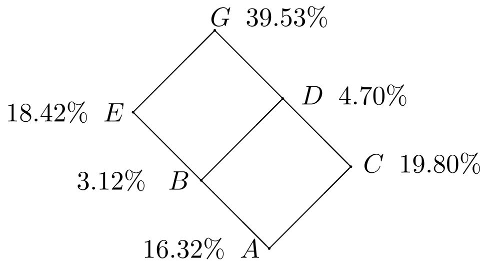

Diagram 4.

Surprisingly the most frequent (39.53%) type of formulas are tautologies. The most infrequent type of formulas (only 3.12%) are those equivalent with the atomic formula a (see definition **??**).

8 Intuitionistic logic of one variable

In this section we show the implicational-negational part of Dummett's linear logic with one propositional variable is identical with the intuitionistic logic of this language. To do this, first we characterize the intuitionistic logic of one variable a with three standard connectives: implication, disjunction and negation. We enumerate some formulas in this language in the following way:

$$F^0 = \neg(a \to a) \tag{60}$$

$$F^1 = a \tag{61}$$

$$F^2 = \neg a \tag{62}$$

$$F^{2n+1} = F^{2n} \vee F^{2n-1} \tag{63}$$

$$F^{2n+2} = F^{2n} \to F^{2n-1} \tag{64}$$

$$\text{for } \ n \geq 1$$

In the set of all formulas we can introduce an equivalence relation $\equiv$ in a common way:

Definition 32 *$\varphi \equiv \psi$ if both $\varphi \to \psi$ and $\psi \to \varphi$ are intuitionistic theorems.*

Every formula from our language $\mathcal{F}^{\{\to,\neg\}}$ falls in to one of the equivalence classes $[F^m]_\equiv$. Therefore up to this equivalence relation on the classes of formulas $[F^m]_\equiv$ give rise to the so-called Rieger - Nishimura lattice $\mathcal{R}$ which is the single-generated free Heyting algebra (see diagram 5).

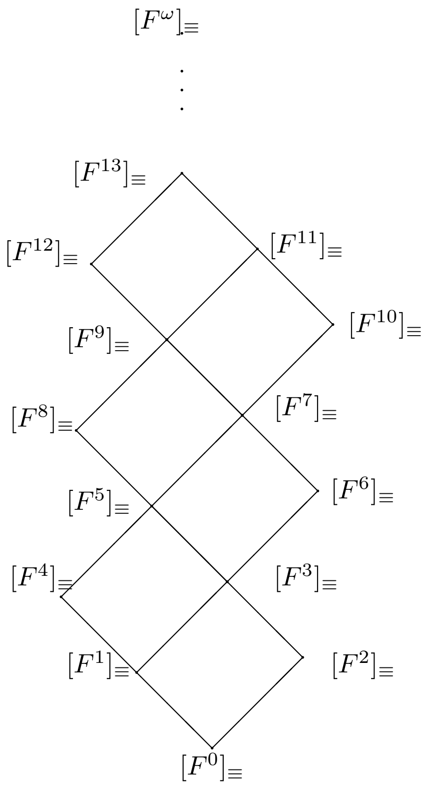

Diagram 5.

Next, we consider an implicational-negational reduct $\mathcal{R}^{\{\to,\neg\}}$ of algebra $\mathcal{R}$.

Lemma 33 *The implicational-negational reduct* $\mathcal{R}^{\{\to,\neg\}}$ *consists of six nonempty classes:*

$$[F^0]_\equiv = [\neg(a \to a)]_\equiv, \tag{65}$$
$$[F^1]_\equiv = [a]_\equiv, \tag{66}$$
$$[F^2]_\equiv = [\neg a]_\equiv, \tag{67}$$
$$[F^4]_\equiv = [\neg a \to a]_\equiv, \tag{68}$$
$$[F^6]_\equiv = [(\neg a \to a) \to a]_\equiv, \tag{69}$$
$$[F^\omega]_\equiv = [a \to a]_\equiv. \tag{70}$$

Proof. The formulas from class $[F^3]_\equiv$ can be obtained as a disjunctions of formulas from classes $[F^1]_\equiv$ and $[F^2]_\equiv$ only. Similarly the formulas from classes $[F^5]_\equiv$, $[F^7]_\equiv$ and $[F^{2n+1}]_\equiv$ for all $n \geq 4$. Hence, in implicational-negational reduct $\mathcal{R}^{\{\rightarrow,\neg\}}$ the classes $[F^{2n+1}]_\equiv$ for all $n \geq 1$, are empty. The emptiness of these classes involves the emptiness of following ones: $[F^8]_\equiv$, $[F^{10}]_\equiv$ and $[F^{2n}]_\equiv$ for all $n \geq 6$, as they are obtained as implications of empty classes also. The class $[F^6]_\equiv$ is nonempty because it contains the formulas which may be obtained as implication of formulas from classes $[F^4]_\equiv$ and $[F^1]_\equiv$ as well. Therefore $[F^6]_\equiv = [(\neg a \rightarrow a) \rightarrow a]_\equiv$. The class $[F^\omega]_\equiv = [a \rightarrow a]_\equiv$ is nonempty as it contains all tautologies. □

Lemma 34 *The implicational-negational part of intuitionistic logic of one variable is identical with the appropriate part of linear logic.*

Proof. It is easy to notice that both the diagrams of implicational-negational reduct $\mathcal{R}^{\{\rightarrow,\neg\}}$ and the Diagram 1 of linear logic of the same language are identical and the following equalities hold:

$$[F^0]_\equiv = A, \tag{71}$$
$$[F^1]_\equiv = B, \tag{72}$$
$$[F^2]_\equiv = C, \tag{73}$$
$$[F^4]_\equiv = E, \tag{74}$$
$$[F^6]_\equiv = D, \tag{75}$$
$$[F^\omega]_\equiv = G. \tag{76}$$

The representatives of appropriate classes are the same and moreover others formulas from these classes stand up in the same way. So, if we make the truth table for classes $[F^0]_\equiv$, $[F^1]_\equiv$, $[F^2]_\equiv$, $[F^4]_\equiv$, $[F^6 =]_\equiv$, $[F^\omega]_\equiv$ from $\mathcal{R}^{\{\rightarrow,\neg\}}$ with $\{\Rightarrow, \sim\}$ it will be the same as Table 1. Hence, the generating functions for appropriate classes are the same also; especially, the generating functions for classes of tautologies G and F^ω. It means the numbers of tautologies of the lenght n both the implicational-negational parts of linear logic of one variable and the intuitionistic one of the same language are equal. Because in general the intuitionistic logic is weaker than the linear one, we conclude that the considered parts of these logic are identical. □

9 Size of intuitionistic logic inside classical

As we know the intuitionistic logic is a proper subset of the classical one. Finally, the result above can be employ to calculate how big is the size of the fragment of intuitionistic logic inside classical logic. The density of implicational-negational part of classical logic of one variable has been calculated in the paper [?]. It has been shown that classical logic of this language admits the limit property ie. there exists a limit of proportions between the number of true formulas and all formulas of the given length. The result has been calculated analytically but numerically it is about 0.4232 . So, the probability of finding a intuitionistic tautology among classical ones is described by the following theorem. We can see that intuitionistic logic is really a dense fragment of classical ones. It is highly probable (probability higher then 93%) to find formulas from intuitionistic logic among random classical tautologies.

Theorem 35 *[Relative density] The relative density of intuitionistic tautologies among classical ones is more then 93 %.*

Proof. Class DG in Diagram 3 is in fact the class of classical tautologies (see [**?**]). We already know asymptotical behavior of classical and intuitionistic logics described by $\lim_{n\to\infty} \frac{|G_n|}{|\mathcal{F}_n^{\{\to,\neg\}}|}$ and $\lim_{n\to\infty} \frac{|DG_n|}{|\mathcal{F}_n^{\{\to,\neg\}}|}$ therefore

$$\lim_{n\to\infty} \frac{|G_n|}{|DG_n|} = \frac{\lim_{n\to\infty} \frac{|G_n|}{|\mathcal{F}_n^{\{\to,\neg\}}|}}{\lim_{n\to\infty} \frac{|DG_n|}{|\mathcal{F}_n^{\{\to,\neg\}}|}} = \frac{0.395305\ldots}{0.44232\ldots} \approx 93\%$$

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