The density of truth in monadic fragments of some intermediate logics

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Abstract

This paper is an attempt to count the proportion of tautologies of some intermediate logics among all formulas. Our interest concentrates especially on Dummett's and Medvedev's logics and their $\{\to, \vee, \neg\}$ fragments over language with one propositional variable.

1 Introduction

Let L be some logical calculus. Let $|T_n|$ be a number of tautologies of length n of that calculus and $|F_n|$ be a number of all formulas of that length. We define the density $\mu(L)$ as:

$$\mu(L) = \lim_{n\to\infty} \frac{|T_n|}{|F_n|}$$

The number $\mu(L)$ if exists, is an asymptotic probability of finding a tautology among all formulas.
This paper is a continuation of research concerning the *density of truth* in different logics. Until now, we were concentrated mostly on classical and intuitionistic logics. Especially, its $\{\to\}$ and $\{\to, \neg\}$ fragments with one variable were investigated (see [5], [9]). It is well known fact that implicational fragments of one variable of intuitionistic and classical logics are the same. Moreover, it is easy to observe that implicational-negational and monadic fragments of every intermediate logic is identical with that same fragment of intuitionistic one. It is already known (see [3]) that the *density* of that logic (and each intermediate as well) is more than 39%. We also know the *density* of implicational-negational and monadic fragment of classical logic. It is about 42%. That gives a very

large amount of intuitionistic tautologies among classical ones in the language consisting of signs of $\{\rightarrow, \neg\}$ and one propositional variable.
It is natural to wish to investigate some important intermediate logics with respect to their densities. We will concentrate our attention on two of them: the Medvedev logic and Dummett's one. To distinguish them we will take their monadic fragments over reacher language consisting of operators $\{\rightarrow, \vee, \neg\}$. Such fragments have models obtained by a simple modification of the Rieger-Nishimura lattice.

2 Intermediate logics

We consider the set of all formulas F built up from one variable p by means of operations $\{\rightarrow, \vee, \neg\}$. Our starting point is Medvedev's logic ML of finite problems. As it is known it is not finitely axiomatizable and might be characterized with help of Kreisel and Putnam's logic [4]. Recall that the logic KP of Kreisel and Putnam is the least intermediate logic with

$$(\neg\alpha \rightarrow (\beta \vee \gamma)) \rightarrow ((\neg\alpha \rightarrow \beta) \vee (\neg\alpha \rightarrow \gamma)) \tag{1}$$

Let $F^{\{\neg\}}$ be the set of formulas defined as follows:

$$\neg\alpha \in F^{\{\neg\}} \quad \Leftrightarrow \quad \alpha \in F; \tag{2}$$

$$\alpha, \beta \in F^{\{\neg\}} \quad \Rightarrow \quad \alpha \rightarrow \beta,\ \alpha \vee \beta \in F^{\{\neg\}}. \tag{3}$$

The characterization is the following:

$$\alpha \in ML \quad \Leftrightarrow \quad (e(\alpha) \in KP, \text{ for every substitution } e : F \rightarrow F^{\{\neg\}})$$

Clearly $KP \subset ML$. The following formula

$$((\neg\neg\alpha \rightarrow \alpha) \rightarrow (\alpha \vee \neg\alpha)) \rightarrow (\neg\neg\alpha \vee \neg\alpha) \tag{4}$$

called Scott's law, belongs to ML and does not belong to KP, see [8].
Hence, the Lindenbaum algebra for the monadic fragment of ML is obtained by dividing the Rieger- Nishimura lattice by the filter generated just by the Scott's law. We will denote this algebra by $\mathcal{M}$. It is consisted of 9 following equivalence classes:

$$\begin{aligned}
A &= [\neg(p \rightarrow p)]_{\equiv}, \\
B &= [p]_{\equiv}, \\
C &= [\neg p]_{\equiv}, \\
D &= [p \vee \neg p]_{\equiv}, \\
E &= [\neg\neg p]_{\equiv}, \\
G &= [\neg\neg p \vee (p \vee \neg p)]_{\equiv}, \\
H &= [\neg\neg p \rightarrow p]_{\equiv}, \\
J &= [(\neg\neg p \vee (p \vee \neg p)) \vee (\neg\neg p \rightarrow p)]_{\equiv}, \\
M &= [p \rightarrow p]_{\equiv},
\end{aligned}$$

and its diagram is as below:

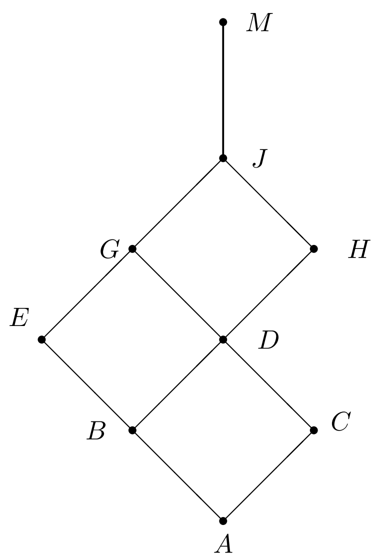

Figure 1.

3 Counting formulas and generating functions

In this section we set up the way of counting formulas with the established length.
way:

$$\begin{aligned} l(p) &= 1 \\ l(\neg\phi) &= 1 + l(\phi) \\ l(\phi \to \psi) &= l(\phi) + l(\psi) + 1 \\ l(\phi \vee \psi) &= l(\phi) + l(\psi) + 1 \end{aligned}$$

length n.
We will also consider some appropriate subclasses of F_n. For example if we have a class $A \subset F$ then $A_n = F_n \cap A$.
class A_n.

Lemma 1 *The number $|F_n|$ of formulas from F_n is given by the recursion:*

$$|F_0| = 0, \ |F_1| = 1, \tag{5}$$

$$|F_n| = |F_{n-1}| + 2\sum_{i=1}^{n-2} |F_i||F_{n-i-1}|. \tag{6}$$

Proof. Any formula of length n for $n > 1$ is either a negation of some formula of length $n-1$ for which the fragment $|F_{n-1}|$ corresponds, or an implication (or disjunction) between some pairs of formulas of lengths i and $n-i-1$, respectively. Therefore the total number of such formulas is $2\sum_{i=1}^{n-2} |F_i||F_{n-i-1}|$. □

The main tool for dealing with asymptotics of sequences of numbers are *generating functions*, see for example [7].

Let $A = (A_0, A_1, A_2, \ldots)$ be a sequence of real numbers. It is in one-to-one correspondence to the formal power series $\sum_{n=0}^{\infty} A_n z^n$. Moreover, considering z as a complex variable, this series, as it is known from the theory of analytic functions, converges uniformly to a function $f_A(z)$ in some open disc $\{z \in \mathcal{C} : |z| < R\}$, where $R \geq 0$ is called its radius of convergence. So, with the sequence A we can associate a complex function $f_A(z)$, called the *ordinary generating function* for A, defined in a neighborhood of 0. This correspondence is one-to-one again (unless $R = 0$), since the expansion of a complex function $f(z)$, analytic in a neighborhood of z_0, into a power series $\sum_{n=0}^{\infty} A_n(z-z_0)^n$ is unique, and moreover, this series is the Taylor series, given by

$$A_n = \frac{1}{n!}\frac{d^n f}{dz^n}(z_0). \tag{7}$$

The above formula is a recursive one. To find a nonrecursive formula for A_n we take advantage from the following result due to Szegö [6] [Thm. 8.4], see also [7] [Thm. 5.3.2]. We need the following much simpler version of the Szegö lemma.

Lemma 2 *Let $v(z)$ be analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$. If $v(z)$ in the vicinity of $z = 1$ has an expansion of the form*

$$v(z) = \sum_{p\geq 0} v_p(1-z)^{\frac{p}{2}}, \tag{8}$$

where $p > 0$, and the branch chosen above for the expansion equals $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = v_1\binom{1/2}{n}(-1)^n + O(n^{-2}). \tag{9}$$

The symbol $[z^n]\{v(z)\}$ stands for the coefficient of z^n in the exponential series expansion of $v(z)$.

Now, we quote (without proof)some theorems which have appeared in [3]. They are the main tools for finding limits of the fraction $\frac{a_n}{b_n}$ when generating functions for sequences a_n and b_n satisfy conditions of simplified Szegö Lemma 2.

Lemma 3 *Suppose two functions $v(z)$ and $w(z)$ satisfy assumptions of simplified Szegö theorem (Lemma 2) i.e. both v and w are analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$. Both $v(z)$ and $w(z)$ in the vicinity of $z = 1$ have expansions of the form*

$$v(z) = \sum_{p \geq 0} v_p (1-z)^{p/2},$$
$$w(z) = \sum_{p \geq 0} w_p (1-z)^{p/2},$$

then the limit of $\frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}}$ exists and is given by the formula:

$$\lim_{n\to\infty} \frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}} = \frac{v_1}{w_1}$$

Theorem 4 *Suppose two functions $v(z)$ and $w(z)$ satisfy assumptions of simplified Szegö theorem (Lemma 2) i.e. both v and w are analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$. Both $v(z)$ and $w(z)$ in the vicinity of $z = 1$ have expansions of the form*

$$v(z) = \sum_{p \geq 0} v_p (1-z)^{p/2},$$
$$w(z) = \sum_{p \geq 0} w_p (1-z)^{p/2},$$

Suppose we have functions $\widetilde{v}$ and $\widetilde{w}$ satisfying $\widetilde{v}(\sqrt{1-z}) = v(z)$ and $\widetilde{w}(\sqrt{1-z}) = w(z)$ then the limit of $\frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}}$ exists and is given by the formula:

$$\lim_{n\to\infty} \frac{[z^n]\{v(z)\}}{[z^n]\{w(z)\}} = \frac{(\widetilde{v})'(0)}{(\widetilde{w})'(0)} \tag{10}$$

4 Gluing of classes

In this section we do some preparations for determining the generating function for the class of tautologies of ML. First, we determine the generating function for the sequence of numbers $|F_n|$.

Lemma 5 *The generating function f for the numbers $|F_n|$ is the following:*

$$f(z) = \frac{1-z}{4} - \frac{\sqrt{z^2 - 10z + 1}}{4}. \tag{11}$$

Proof. The recurrence formula $|F_n| = |F_{n-1}| + 2\sum_{i=1}^{n-2} |F_i||F_{n-i-1}|$ becomes the equality

$$f(z) = zf(z) + 2f^2(z) + z \tag{12}$$

since the recursion fragment $\sum_{i=1}^{n-2} |F_i||F_{n-i-1}|$ corresponds exactly to multiplication of power series. The term $|F_{n-1}|$ corresponds to the function $zf(z)$. The linear term z corresponds to the first non-zero coefficient in the power series of f. Solving the equation we get two possible solutions: $f(z) = \frac{1-z}{4} - \frac{\sqrt{z^2-10z+1}}{4}$ or $f(z) = \frac{1-z}{4} + \frac{\sqrt{z^2-10z+1}}{4}$. We choose the first one, since it corresponds to the boundary condition $f(0) = 0$ (see equation (5)). □

To find the generating functions for other classes of formulas it will be useful to have written the truth-table for the operations $\{\to, \vee, \neg\}$ of the algebra $\mathcal{M}$ presented in Figure 1.

$\to$	A	B	C	D	E	G	H	J	M	$\neg$
A	M	M	M	M	M	M	M	M	M	M
B	C	M	C	M	M	M	M	M	M	C
C	E	E	M	M	E	M	M	M	M	E
D	A	E	C	M	E	M	M	M	M	A
E	C	H	C	H	M	M	H	M	M	C
G	A	B	C	H	E	M	H	M	M	A
H	A	E	C	G	E	G	M	M	M	A
J	A	B	C	D	E	G	H	M	M	A
M	A	B	C	D	E	G	H	J	M	A

Table 1.

$\vee$	A	B	C	D	E	G	H	J	M
A	A	B	C	D	E	G	H	J	M
B	B	B	D	D	E	G	H	J	M
C	C	D	C	D	G	G	H	J	M
D	D	D	D	D	G	G	H	J	M
E	E	E	G	G	E	G	J	J	M
G	G	G	G	G	G	G	J	J	M
H	H	H	H	H	J	J	H	J	M
J	J	J	J	J	J	J	J	J	M
M	M	M	M	M	M	M	M	M	M

Table 2.

From these tables we can read the way of creating each tautology and any formula from any other class. For example the formulas from the class A are built up as implications between formulas from the classes D, G, H, J, M and A, negations of formulas from D, G, H, J, M and disjunctions of formulas from class A. But if we want to count the quantities of that formulas we would have to know the way of creating formulas from the other needed classes. This provides to obtaining a system of nine recurrent equations and consequently a system of nine non-linear equations with nine unknown generating functions. To avoid such complication we will consider step by step much simpler algebras obtained from the main quotient algebra $\mathcal{M}$.

First let us take the filter $\{J, M\}$ and consider the new algebra $\mathcal{M}/_{\{J,M\}}$. Its diagram and the appropriate truth - tables are presented below.

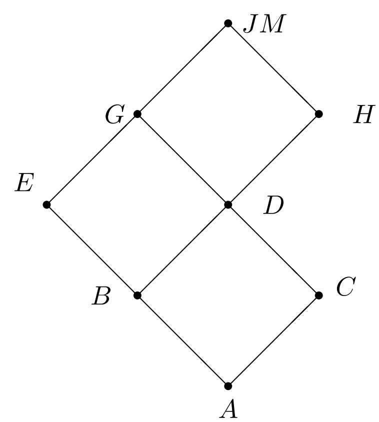

Figure 2

where

$$JM = J \cup M$$

We will repeat this abbreviation for other classes.

$\rightarrow$	A	B	C	D	E	G	H	JM	$\neg$
A	JM	JM	JM	JM	JM	JM	JM	JM	JM
B	C	JM	C	JM	JM	JM	JM	JM	C
C	E	E	JM	JM	E	JM	JM	JM	E
D	A	E	C	JM	E	JM	JM	JM	A
E	C	H	C	H	JM	JM	H	JM	C
G	A	B	C	H	E	JM	H	JM	A
H	A	E	C	G	E	G	JM	JM	A
JM	A	B	C	D	E	G	H	JM	A

Table 3.

$\vee$	A	B	C	D	E	G	H	JM
A	A	B	C	D	E	G	H	JM
B	B	B	D	D	E	G	H	JM
C	C	D	C	D	G	G	H	JM
D	D	D	D	D	G	G	H	JM
E	E	E	G	G	E	G	JM	JM
G	G	G	G	G	G	G	JM	JM
H	H	H	H	H	JM	JM	H	JM
JM	JM	JM	JM	JM	JM	JM	JM	JM

Table 4.

Now, we take the filter $\{H, J, M\}$ and consider the new algebra $\mathcal{M}/_{\{H,J,M\}}$. Its diagram and the appropriate truth - tables are the following:

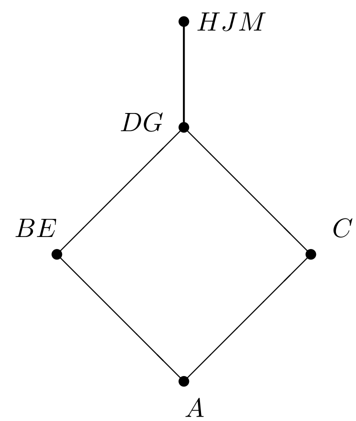

Figure 3.

$\to$	A	BE	C	DG	HJM	$\neg$
A	HJM	HJM	HJM	HJM	HJM	HJM
BE	C	HJM	C	HJM	HJM	C
C	BE	BE	HJM	HJM	HJM	BE
DG	A	BE	C	HJM	HJM	A
HJM	A	BE	C	DG	HJM	A

Table 5.

$\vee$	A	BE	C	DG	HJM
A	A	BE	C	DG	HJM
BE	BE	BE	DG	DG	HJM
C	C	DG	C	DG	HJM
DG	DG	DG	DG	DG	HJM
HJM	HJM	HJM	HJM	HJM	HJM

Table 6.

We also divide the algebra $\mathcal{M}$ by the filter $L = \{G, J, M\}$:

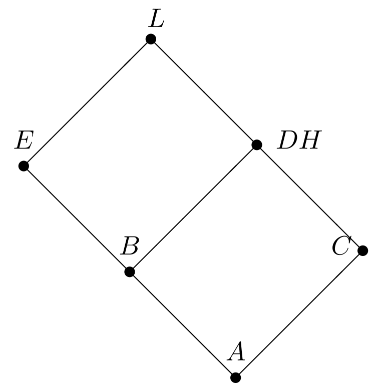

Figure 4.

The truth tables are the following:

$\to$	A	B	C	DH	E	L	$\neg$
A	L	L	L	L	L	L	L
B	C	L	C	L	L	L	C
C	E	E	L	L	E	L	E
DH	A	E	C	L	E	L	A
E	C	DH	C	DH	L	L	C
L	A	B	C	DH	E	L	A

Table 7.

$\vee$	A	B	C	DH	E	L
A	A	B	C	DH	E	L
B	B	B	DH	DH	E	L
C	C	DH	C	DH	L	L
DH	DH	DH	DH	DH	L	L
E	E	E	L	L	E	L
L	L	L	L	L	L	L

Table 8.

Lemma 6 *The matrix described in Tables 7 and 8 is a matrix for the linear Dummett's logic of implication, disjunction and negation with one variable* [1].

Proof. Proof of that fact is poorly semantical, analogous to the one of Lemma 7 in [3].
The next divisions are the following - we divide the algebra $\mathcal{M}$ by the filters $\{E, G, J, M\}$ and $K = \{D, G, H, J, M\}$ and obtain:

[1] Linear calculus LC was studied in [1] by Dummett. Syntactically it is obtained by adding the axiom $(p \to q) \vee (q \to p)$ to axioms of intuitionistic logic.

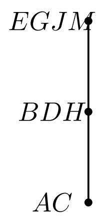

Figure 5.

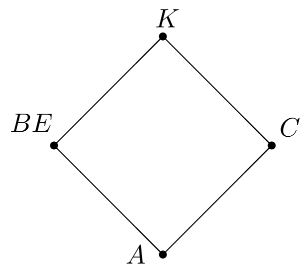

Figure 6.

The operations $\{\to, \vee, \neg\}$ in the new algebras are given by the following truth-tables:

$\to$	AC	BDH	$EGJM$	$\neg$
AC	$EGJM$	$EGJM$	$EGJM$	$EGJM$
BDH	AC	$EGJM$	$EGJM$	AC
$EGJM$	AC	BDH	$EGJM$	AC

Table 9.

$\vee$	AC	BDH	$EGJM$
AC	AC	BDH	$EGJM$
BDH	BDH	BDH	$EGJM$
$EGJM$	$EGJM$	$EGJM$	$EGJM$

Table 10.

$\to$	A	BE	C	K	$\neg$
A	K	K	K	K	K
BE	C	K	C	K	C
C	BE	BE	K	K	BE
K	A	BE	C	K	A

Table 11.

$\vee$	A	BE	C	K
A	A	BE	C	K
BE	BE	BE	K	K
C	C	K	C	K
K	K	K	K	K

Table 12.

As we can observe, the first truth table describes operations in the Gödel 3 valued matrix, while the second one is a matrix of all valuations associated with the standard classical logic of one variable.

Lemma 7 *The matrix described in Tables 11 and 12 is a matrix for the classical logic of implication, disjunction and negation with one variable.*

Proof. Proof of that fact is analogous to the one of Lemma 12 in [3]. □

The last two divisions are the following: we take two filters $N = \{C, D, G, H, J, M\}$ and $P = \{B, D, E, G, H, J, M\}$. The new quotient algebras $\mathcal{M}/_N$ and $\mathcal{M}/_P$ have the following diagrams:

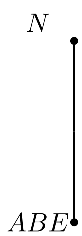

Figure 7.

Figure 8.

The operations $\{\to, \vee, \neg\}$ are characterized by the following truth-tables:

$\to$	ABE	N	$\neg$
ABE	N	N	N
N	ABE	N	ABE

Table 13.

$\to$	ABE	N
ABE	ABE	N
N	N	N

Table 14.

$\to$	AC	P	$\neg$
AC	P	P	P
P	AC	P	AC

Table 15.

$\to$	AC	P
AC	AC	P
P	P	P

Table 16.

5 Calculating generating functions

Now,we are ready to deal with generating functions. We start with analyzing the algebra $\mathcal{M}/_N$.

Lemma 8 *The numbers $|ABE_n|$ are given by the following mutual recursions:*

$$|ABE_0| = 0, \quad |ABE_1| = 1, \tag{13}$$

$$|ABE_n| = \sum_{i=1}^{n-2} |F_i||ABE_{n-i-1}| + |N_{n-1}|, \tag{14}$$

Proof. From Tables 13 and 14 we see that formulas from class ABE can be obtained as implications of formulas from classes N and ABE - this gives us the recurrence $\sum_{i=1}^{n-2} |N_i||ABE_{n-i-1}|$, disjunctions of formulas from class ABE - hence we have $\sum_{i=1}^{n-2} |ABE_i||ABE_{n-i-1}|$, or negations of ones from the class N, which gives the part $|N_{n-1}|$. Hence we have:

$$|ABE_n| = \sum_{i=1}^{n-2} |N_i||ABE_{n-i-1}| + \sum_{i=1}^{n-2} |ABE_i||ABE_{n-i-1}| + |N_{n-1}|.$$

From disjointness of considered classes we have $|ABE_i| + |N_i| = |F_i|$ and hence we have (14). Because $p \in ABE$ then we have (13). □

Lemma 9 *The generating function f_{ABE} for sequence of numbers $|ABE_n|$ is:*

$$f_{ABE}(z) = \frac{zf(z) + z}{1 - f(z) + z} \tag{15}$$

$$\textit{where} \;\; f(z) = \frac{1-z}{4} - \frac{\sqrt{z^2 - 10z + 1}}{4}$$

Proof. The part $\sum_{i=1}^{n-2} |F_i||ABE_{n-i-1}|$ of the recurrence (14) corresponds to multiplication of power series and then gives multiplication of generating functions: $f(z)f_{ABE}(z)$. The term $|N_{n-1}| = |F_i| - |ABE_i|$ corresponds to the function $z(f(z) - f_{ABE}(z))$. Hence the recurrence (14) gives the following equality between the appropriate generating functions:

$$f_{ABE}(z) = f(z)f_{ABE}(z) + z(f(z) - f_{ABE}(z)) + z \tag{16}$$

The linear term z in (16) corresponds to the first non-zero coefficient in the power series of f_{ABE}. A basic transformation gives us (15). □
Analogously we determine the appropriate recurrence and generating function for the numbers $|AC_n|$.

Lemma 10 *The numbers $|AC_n|$ are given by the following mutual recursions:*

$$|AC_0| = |AC_1| = 0, \quad |AC_2| = 1, \tag{17}$$

$$|AC_n| = \sum_{i=1}^{n-2} |F_i||AC_{n-i-1}| + |P_{n-1}|, \tag{18}$$

Proof. The above recurrence follows from Tables 15 and 16 analogously to the one in Lemma 8. □

Lemma 11 *The generating function f_{AC} for sequence of numbers $|AC_n|$ is the following:*

$$f_{AC}(z) = \frac{zf(z)}{1 - f(z) + z} \quad \text{where } f(z) = \frac{1-z}{4} - \frac{\sqrt{z^2 - 10z + 1}}{4} \tag{19}$$

Proof. The recurrence (18) can be translated into following equation:

$$f_{AC}(z) = f(z)f_{AC}(z) + z(f(z) - f_{AC}(z)) \tag{20}$$

which gives us (19). There is no linear term in (20) because $p \notin AC$. □

In the same manner we can determine generating functions connected with other classes of formulas. For simplicity of notation we will omit the argument (z) in all the functions which will arrive hereafter. Now, we determine the generating function for the class of classical tautologies K.

Lemma 12 *The generating function f_K for the numbers $|K_n|$ is the following:*

$$f_K = \frac{1}{16}\left(4 - 4z + \frac{12f(-1+f+z) + 12z}{-1+f-z} - S\right), \tag{21}$$

where

$$f(z) = \frac{1-z}{4} - \frac{\sqrt{z^2-10z+1}}{4}$$

$$S = \frac{\sqrt{2}\sqrt{8 - z + 20z^2 + 31z^3 + 8z^4 + f(-15 - 31z - z^2 + 15z^3)}}{1 - f + z}.$$

Proof. From Tables 11 and 12 we have the following recurrence for the numbers $|A_n|$ hold:

$$|A_0| = |A_1| = |A_2| = |A_3| = 0, \quad |A_4| = 1, \tag{22}$$

$$|A_n| = \sum_{i=1}^{n-2} |K_i||A_{n-i-1}| + \sum_{i=1}^{n-2} |A_i||A_{n-i-1}| + |K_{n-1}|, \tag{23}$$

This recurrence can be translated into equation:

$$f_A = f_K f_A + f_A^2 + z f_K \tag{24}$$

with two unknown functions f_A and f_K.
From the other side, from definitions of classes AC, ABE and K we have:

$$f_A = f_{AC} + f_K + f_{ABE} - f \tag{25}$$

From the system of equations:

$$\begin{cases} (24) \\ (25) \end{cases}$$

we obtain the quadratic equation:

$$2f_K^2 + 3f_K(f_{AC} + f_{ABE} - f) + (f_{AC} + f_{ABE} - f)^2 - f_{AC} - f_{ABE} + f = 0$$

After solving it with the boundary condition $f_K(0) = 0$ and taking into consideration the equalities (19), (15), (11) and intensive simplification we get (21). □

The next important generating function is the one of class of linear tautologies L (see Figure 4). To determine it we must first consider Figure 5 and determine the generating function for the class of formulas BDH.

Lemma 13 *The generating function f_{BDH} for the sequence of numbers $|BDH_n|$ is:*

$$\begin{aligned} f_{BDH} &= \frac{2z(1-f+z)}{-3f-5zf+z+2}, \qquad (26) \\ &\textit{where} \\ f(z) &= \frac{1-z}{4} - \frac{\sqrt{z^2-10z+1}}{4} \end{aligned}$$

Proof. It follows easily from Tables 9 and 10 that the numbers $|BDH_n|$ are given by the following recursion:

$$\begin{aligned} |BDH_0| &= 0, \quad |BDH_1| = 1, \\ |BDH_n| &= \sum_{i=1}^{n-2} |EGJM_i||BDH_{n-i-1}| + 2\sum_{i=1}^{n-2} |BDH_i||AC_{n-i-1}| + \\ &\quad + \sum_{i=1}^{n-2} |BDH_i||BDH_{n-i-1}|. \qquad (27) \end{aligned}$$

The above recurrence leads to the equality:

$$f_{BDH} = f_{EGJM} f_{BDH} + 2f_{BDH} f_{AC} + f_{BDH}^2 + z \qquad (28)$$

From disjointness of the appropriate classes we have that:

$$f_{EGJM} = f - f_{BDH} - f_{AC}. \qquad (29)$$

and after application (29) to the equation (28) we obtain as follows:

$$f_{BDH} = \frac{z}{1 - f_{AC} - f} \qquad (30)$$

which after a suitable simplification gives us (26). □

The next needed function is the function f_B. We take into consideration Figure 4.

Lemma 14 *The generating function f_B for the sequence of numbers $|B_n|$ is the following:*

$$
\begin{aligned}
f_B &= \frac{12 + S + 4z + U - 16X - 16\sqrt{-8z + (-1 - 2f + Y + X)^2}}{64}, \quad (31)\\
&\quad \textit{where}\\
f(z) &= \frac{1-z}{4} - \frac{\sqrt{z^2 - 10z + 1}}{4}\\
S &= \frac{\sqrt{2}\sqrt{8 - z + 20z^2 + 31z^3 + 8z^4 + f(-15 - 31z - z^2 + 15z^3)}}{1 - f + z}.\\
U &= \frac{-8z - f(12f - 2S) + 2(4 - 4z^2 - S(1+z))}{-1 + f - z}\\
X &= \frac{2z(1 - f + z)}{-2 - z + f(3 + 5z)}\\
Y &= \frac{-14z - 58fz + 3(-4 + S - f(2 + S) + Sz + 4z^2)}{16(-1 + f - z)} \qquad (32)
\end{aligned}
$$

Proof. From Tables 7 and 8 we get the following recurrence concerning the numbers $|B_n|$:

$$
\begin{aligned}
|B_0| &= 0, \quad |B_1| = 1,\\
|B_n| &= \sum_{i=1}^{n-2} |L_i||B_{n-i-1}| + 2\sum_{i=1}^{n-2} |B_i||A_{n-i-1}| +\\
&\quad + \sum_{i=1}^{n-2} |B_i||B_{n-i-1}|. \qquad (33)
\end{aligned}
$$

The above recurrence leads to the equality:

$$f_B = f_L f_B + 2 f_B f_A + f_B^2 + z\,. \qquad (34)$$

We also have:

$$
\begin{aligned}
&f_L = f_{EGJM} - f_E, \quad f_{EGJM} = f - f_{BDH} - f_{AC} \quad f_E = f_{BE} - f_B,\\
&f_{BE} = f - f_{AC} - f_K, \quad f_A = f_{AC} + f_K + f_{ABE} - f\,.
\end{aligned}
$$

After a suitable substitution and simplification we get:

$$f_L = f_K + f_B - f_{BDH}. \qquad (35)$$

We apply (35) to (34) and after simplification obtain the following quadratic equation:

$$2f_B^2 + f_B(2f_{AC} + 3f_K + 2f_{ABE} - 2f - f_{BDH} - 1) + z = 0. \qquad (36)$$

Solving (36) with the boundary condition $f_B(0) = 0$ and a suitable substitution of (19), (21), (15) and (26) give us (31). □

Now, we are ready to determine the generating function of linear tautologies. Let us take advantage of Lemma 14 and (35).

Corollary 15 *The generating function f_L for the sequence of numbers $|L_n|$ is the following:*

$$f_L = \frac{1}{64}\Bigg(28 - 12z - 3S + U + 48X + 48\frac{f(-1+f+z)+z}{-1+f-z} - 16\sqrt{-8z + (-1-2f+Y+X)^2}\Bigg), \quad (37)$$

where

$$f = \frac{1-z}{4} - \frac{\sqrt{z^2-10z+1}}{4},$$

$$S = \frac{\sqrt{2}\sqrt{8 - z + 20z^2 + 31z^3 + 8z^4 + f(-15 - 31z - z^2 + 15z^3)}}{1-f+z}, \quad (38)$$

$$U = \frac{-8z - f(12f - 2S) + 2(4 - 4z^2 - S(1+z))}{-1+f-z}, \quad (39)$$

$$X = \frac{2z(1-f+z)}{-2-z+f(3+5z)}, \quad (40)$$

$$Y = \frac{-14z - 58fz + 3(-4 + S - f(2+S) + Sz + 4z^2)}{16(-1+f-z)}. \quad (41)$$

The next step will concern the lattice presented in Figure 3. We notice that:

Lemma 16 *The generating function f_{DG} for the numbers $|DG_n|$ is the following:*

$$f_{DG} = \frac{T}{8(-1+f-z)} * \\ * \frac{(T+16z)}{-20 + S(1+z-f) + z(6+4z) + f(30+50z)} \quad (42)$$

where

$$f = \frac{1-z}{4} - \frac{\sqrt{z^2-10z+1}}{4}$$

$$S = \frac{\sqrt{2}\sqrt{8 - z + 20z^2 + 31z^3 + 8z^4 + f(-15 - 31z - z^2 + 15z^3)}}{1-f+z}$$

$$T = -4 + S - 2z + Sz + 4z^2 + f(6 - S + 10z)$$

Proof. From Tables 5 and 6 we get the following recurrence concerning the numbers $|DG_n|$:

$$|DG_0| = |DG_1| = |DG_2| = |DG_3| = 0, \quad |DG_4| = 1,$$

$$|DG_n| = \sum_{i=1}^{n-2} |HJM_i||DG_{n-i-1}| + 2\sum_{i=1}^{n-2}(|A_i| + |BE_i| + |C_i|)|DG_{n-i-1}| +$$

$$+\sum_{i=1}^{n-2}|DG_i||DG_{n-i-1}|+2\sum_{i=1}^{n-2}|C_i||BE_{n-i-1}|. \tag{43}$$

The above recurrence gives the equality:

$$f_{DG}=f_{HJM}f_{DG}+2(f_A+f_{BE}+f_C)f_{DG}+f_{DG}^2+2f_Cf_{BE}. \tag{44}$$

Because $f_{HJM}=f_K-f_{DG}$, $f_A+f_{BE}+f_C=f-f_K$, $f_C=f-f_K-f_{ABE}$ and $f_{BE}=f-f_{AC}-f_K$ then we obtain the following linear equation with respect the function f_{DG}:

$$f_{DG}(2f-f_K-1)+2(f-f_K-f_{ABE})(f-f_{AC}-f_K)=0. \tag{45}$$

This equation after suitable substitution and simplification give us (42). □

Now, we are very close to achieve the main goal of that paper. All that is left for us to do, is to determine two generating functions - one connected with Figure 2 (we will choose f_D) and the last one which is the generating function for the class of tautologies of Medvedev's calculus. Let us consider Figure 2.

Lemma 17 *The generating function f_D for the numbers $|D_n|$ is the following:*

$$\begin{aligned} f_D &= \frac{1}{4}\Big(1-2f_{AC}-f_L+f_{DG}-2f_B- \\ &\quad \sqrt{16(f_{ABE}-f+f_K)f_B+(-1+2f_{AC}+f_L-f_{DG}+2f_B)^2}\Big) \end{aligned} \tag{46}$$

where functions f, f_{ABE}, f_{AC}, f_K, f_B, f_L, f_{DG} are defined by (11), (15), (19), (21), (31), (37) and (42).

Proof. Tables 3 and 4 give us the following recurrence:

$$\begin{aligned} |D_0| &= |D_1|=|D_2|=|D_3|=0, \quad |D_4|=1, \\ |D_n| &= 2(\sum_{i=1}^{n-2}|B_i||C_{n-i-1}|+\sum_{i=1}^{n-2}|D_i|(|A_{n-i-1}|+|C_{n-i-1}|+|B_{n-i-1}|))+ \\ &\quad \sum_{i=1}^{n-2}|JM_i||D_{n-i-1}|+\sum_{i=1}^{n-2}|D_i||D_{n-i-1}|. \end{aligned} \tag{47}$$

This recurrence can be translated as follows:

$$f_D=f_{JM}f_D+2(f_Bf_C+(f_A+f_C+f_B)f_D)+f_D^2. \tag{48}$$

We know also that $f_{JM}=f_L-f_{DG}+f_D$, $f_C=f-f_K-f_{ABE}$ and $f_A+f_C+f_B=f_{AC}+f_B$.

Therefore we obtain the following quadratic equation with respect to the function f_D:

$$2f_D^2+f_D(f_L-f_{DG}+2(f_{AC}+f_B)-1)+2f_B(f-f_K-f_{ABE})=0. \tag{49}$$

After solving it with the condition $f_D(0) = 0$ we get (46). For simplicity we have expressed f_D in terms of before determined generating functions. □

Now, we are prepare to determine the generating function characterizing tautologies of Medvedev's logic. To do that we choose the class J_n because it is the one which distinguishes the algebras $\mathcal{M}$ and $\mathcal{M}/_{\{JM\}}$.

Lemma 18 *The generating function f_J for the numbers $|J_n|$ is the following:*

$$f_J = \frac{2(f - f_{BDH} - f_{AC} - f_L - f_D + f_{DG})(f_K - f_L - f_D)}{1 + f_D - 2f + f_L - f_{DG}} \quad (50)$$

where functions f, f_{AC}, f_K, f_{BDH}, f_L, f_{DG}, f_D are defined by (11), (19), (21), (26), (37), (42) and (46).

Proof. Tables 1 and 2 give us the following recurrence:

$$\begin{aligned} |J_0| &= ... = |J_8| = 0, \quad |J_9| = 1, \\ |J_n| &= \sum_{i=1}^{n-2} |M_i||J_{n-i-1}| + 2(\sum_{i=1}^{n-2} |H_i|(|E_{n-i-1}| + |G_{n-i-1}|) + \\ &\quad \sum_{i=1}^{n-2} |J_i|(|F_{n-i-1} - |J_{n-i-1}| - |M_{n-i-1}|)) + \sum_{i=1}^{n-2} |J_i||J_{n-i-1}|. \end{aligned} \quad (51)$$

This recurrence can be translated as follows:

$$f_J = f_M f_J + 2(f_H(f_E + f_G) + f_J(f - f_J - f_M)) + f_J^2. \quad (52)$$

After taking advantage of the following equalities $f_M = f_{JM} - f_J$, $f_{JM} = f_L + f_D - f_{DG}$, $f_E + f_G = f - f_{BDH} - f_{AC} - f_{JM}$ $f_H = f_K - f_L - f_D$ we obtain a linear equation which after solving gives us (50). □

Corollary 19 *The generating function f_M for the sequence of numbers $|M_n|$ is the following:*

$$f_M = f_L + f_D - f_{DG} - f_J \quad (53)$$

where functions f_L, f_{DG}, f_D and f_J are defined by (37), (42), (46) and (50).

6 Counting asymptotic densities

In this section we do some calculations concerning singularities of the investigated generating functions. First, let us observe that:

Lemma 20 *$z_0 = 5 - 2\sqrt{6}$ is the only singularity of f, f_K, f_L and f_M located in $|z| \le 5 - 2\sqrt{6}$.*

Proof. It is easy to observe the function $f(z)$ has only singularities at $z = 5-2\sqrt{6}$ and $z = 5+2\sqrt{6}$. To make sure the functions f_K, f_L and f_M have the nearest one at $z = 5-2\sqrt{6}$, we had to solve the following complicated equations:

$$\begin{aligned} -1+f-z &= 0 \\ S &= 0 \\ -8z+(-1-2f+Y+X)^2 &= 0 \\ -20+S(1+z-f)+z(6+4z+f(30+50z) &= 0 \\ 16(f_{ABE}-f+f_K)f_B+(-1+2f_{AC}+f_L-f_{DG}+2f_B)^2 &= 0 \\ -3f-5zf+z+2 &= 0 \\ 1+f_D-2f+f_L-f_{DG} &= 0 \end{aligned}$$

where functions f, f_{AC}, f_{ABE}, f_K, f_{DG}, f_B, f_L, f_D, X, Y, S are defined by (11), (19), (15), (21), (42), (31), (37), (46), (40), (41), (38). To do that we had to use extensively *the Mathematica* package and it occurred that all solutions which are different from $z = 5-2\sqrt{6}$ are situated outside the disc $|z| \leq 5-2\sqrt{6}$. □

To apply the Szegö Lemma we have to have functions which are analytic in the open disc $|z| < 1$, and the nearest singularity is at $z_0 = 1$. For that purpose we are going to calibrate functions f, f_K, f_L and f_M in the following way:

$$\begin{aligned} \widehat{f}(z) &= f\left(\tfrac{z}{5-2\sqrt{6}}\right) & \qquad \widehat{f_K}(z) &= f_K\left(\tfrac{z}{5-2\sqrt{6}}\right) \\ \widehat{f_L}(z) &= f_L\left(\tfrac{z}{5-2\sqrt{6}}\right) & \qquad \widehat{f_M}(z) &= f_M\left(\tfrac{z}{5-2\sqrt{6}}\right). \end{aligned}$$

It is not essential for our task to express the above functions in explicit forms. We only note that the relations between power series of the appropriate functions are such as $[z^n]\{f(z)\} = \left([z^n]\{\widehat{f}(z)\}\right)(5-2\sqrt{6})^n$.

Corollary 21 *$z_0 = 1$ is the only singularity of $\widehat{f}$, $\widehat{f_K}$, $\widehat{f_L}$ and $\widehat{f_M}$ located in $|z| \leq 1$.*

Theorem 22 *Expansions of functions $\widehat{f}$ and $\widehat{f_K}$ in a neighborhood of $z = 1$ are as follows:*

$$\widehat{f}(z) = f_0 + f_1\sqrt{1-z} + ...$$
$$\widehat{f_K}(z) = k_0 + k_1\sqrt{1-z} + ...$$

where

$$f_0 = \frac{-4+2\sqrt{6}}{4}, \quad f_1 = -\frac{1}{4}\sqrt{-48+20\sqrt{6}}, \quad k_0 \approx 0.06468\ldots, \quad k_1 \approx -0.16307\ldots$$

Proof. The above coefficients were found using *the Mathematica* package. □

Analogously we count the appropriate coefficients of generating functions of linear tautologies and Medvedev's ones.

Theorem 23 *Expansions of functions* $\widehat{f_L}$ *and* $\widehat{f_M}$ *in a neighborhood of* $z = 1$ *are as follows:*

$$\widehat{f_L}(z) = l_0 + l_1\sqrt{1-z} + ...$$
$$\widehat{f_M}(z) = m_0 + m_1\sqrt{1-z} + ...$$

where

$$l_0 \approx 0.05534\ldots, \quad l_1 = -0.14583\ldots, \quad m_0 \approx 0.054511\ldots, \quad m_1 \approx -0.14279\ldots$$

Now, we can calculate the density of implicational-disjunctional-negational parts of classical, linear and Medvedev's logic of one variable. By application of the Szegö lemma, Lemma 3 and Theorem 4 we get as follows:

Theorem 24

$$\begin{aligned}\mu(K) &= \lim_{n\to\infty}\frac{|K_n|}{|F_n|} = \lim_{n\to\infty}\frac{(k_1\binom{1/2}{n}(-1)^n + O(n^{-2}))(5-2\sqrt{6})^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))(5-2\sqrt{6})^n}\\ &= \lim_{n\to\infty}\frac{k_1}{f_1}(1+o(1)) = \frac{k_1}{f_1} \approx 65.56\%\end{aligned}$$

Theorem 25

$$\begin{aligned}\mu(L) &= \lim_{n\to\infty}\frac{|L_n|}{|F_n|} = \lim_{n\to\infty}\frac{(l_1\binom{1/2}{n}(-1)^n + O(n^{-2}))(5-2\sqrt{6})^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))(5-2\sqrt{6})^n}\\ &= \lim_{n\to\infty}\frac{l_1}{f_1}(1+o(1)) = \frac{l_1}{f_1} \approx 58.63\%\end{aligned}$$

Theorem 26

$$\begin{aligned}\mu(M) &= \lim_{n\to\infty}\frac{|M_n|}{|F_n|} = \lim_{n\to\infty}\frac{(m_1\binom{1/2}{n}(-1)^n + O(n^{-2}))(5-2\sqrt{6})^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))(5-2\sqrt{6})^n}\\ &= \lim_{n\to\infty}\frac{m_1}{f_1}(1+o(1)) = \frac{m_1}{f_1} \approx 57.41\%\end{aligned}$$

The results presented above can be compared with analogous one concerning the density of implicational-negational parts of linear and classical logics with one variable. As it was said in **Introduction** they amount to 39% and 42%. Then one can notice the operation of disjunction is in fact a truth 'makers'. From the other side we can see that the quantitative difference between considered fragments of Medvedev's and linear calculi are almost imperceptibly.

Theorem 27 *The relative probability of finding a tautology of Medvedev's logic among linear ones is about 98 %.*

Proof. From the known asymptotics $\lim_{n\to\infty} \frac{|M_n|}{|F_n|}$ and $\lim_{n\to\infty} \frac{|L_n|}{|F_n|}$ we get

$$\lim_{n\to\infty} \frac{|M_n|}{|L_n|} = \frac{\lim_{n\to\infty} \frac{|M_n|}{|F_n|}}{\lim_{n\to\infty} \frac{|L_n|}{|F_n|}} = \frac{0.5741\ldots}{0.5863\ldots} \approx 98\%.$$

□

Finally, the above results can be employed to calculate the size of fragment of Dummett's logics inside classical one.

Theorem 28 *The relative probability of finding a linear tautology among classical ones is more then 89 %.*

Proof. We already know asymptotics $\lim_{n\to\infty} \frac{|L_n|}{|F_n|}$ and $\lim_{n\to\infty} \frac{|K_n|}{|F_n|}$ therefore

$$\lim_{n\to\infty} \frac{|L_n|}{|K_n|} = \frac{\lim_{n\to\infty} \frac{|L_n|}{|F_n|}}{\lim_{n\to\infty} \frac{|K_n|}{|F_n|}} = \frac{0.5863\ldots}{0.6556\ldots} \approx 89\%.$$

□

Let us compare the last result with the one concerning the $\{\to, \neg\}$ fragment of the monadic linear and classical logics. In [2] it is shown the relative probability of finding a linear tautology among classical ones of such language is more than 93%. This can be commented that the operations of disjunctions (linear and classical) play an important role in distinction between the considered logics.

References

[1] M. Dummet, *A propositional calculus with a denumerable matrix*, JSL, vol.24, 1954, pp.96-107.

[2] Kostrzycka, Z. *On the density of truth in Dummet's logic*, Bulletin of the Section of Logic, 32(1/2): 43-55, 2003.

[3] Kostrzycka, Z. and Zaionc, M. *Statistics of intuitionistic versus classical logics*, Studia Logica, 76(3): 307-328, 2004.

[4] Levin, L. *Some syntactical theorems on the Yu. T. Medvedev's calculus of finite problems*, Doklady Academii Nauk SSSR (Soviet Mathematics, Doklady), 245: 1051-1054, 1979.

[5] M. Moczurad, J. Tyszkiewicz, M. Zaionc "Statistical properties of simple types " Mathematical Structures in Computer Science, vol 10 (2000), pp 575-594.

[6] Szegö, G. (1975) Orthogonal polynomials. Fourth edition.*AMS, Colloquium Publications*, **23**, Providence.

[7] Wilf, H.S. (1994) Generating functionology. Second edition.*Academic Press*, Boston.

[8] Wojtylak, P. *On a Problem of H. Friedman and its Solution by T. Prucnal.* Report on Mathematical Logic, 38: 69-86, 2004.

[9] M. Zaionc, *On the asymptotic density of tautologies in logic of implication and negation*, Reports on Mathematical Logic, vol. 39, 2004.

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